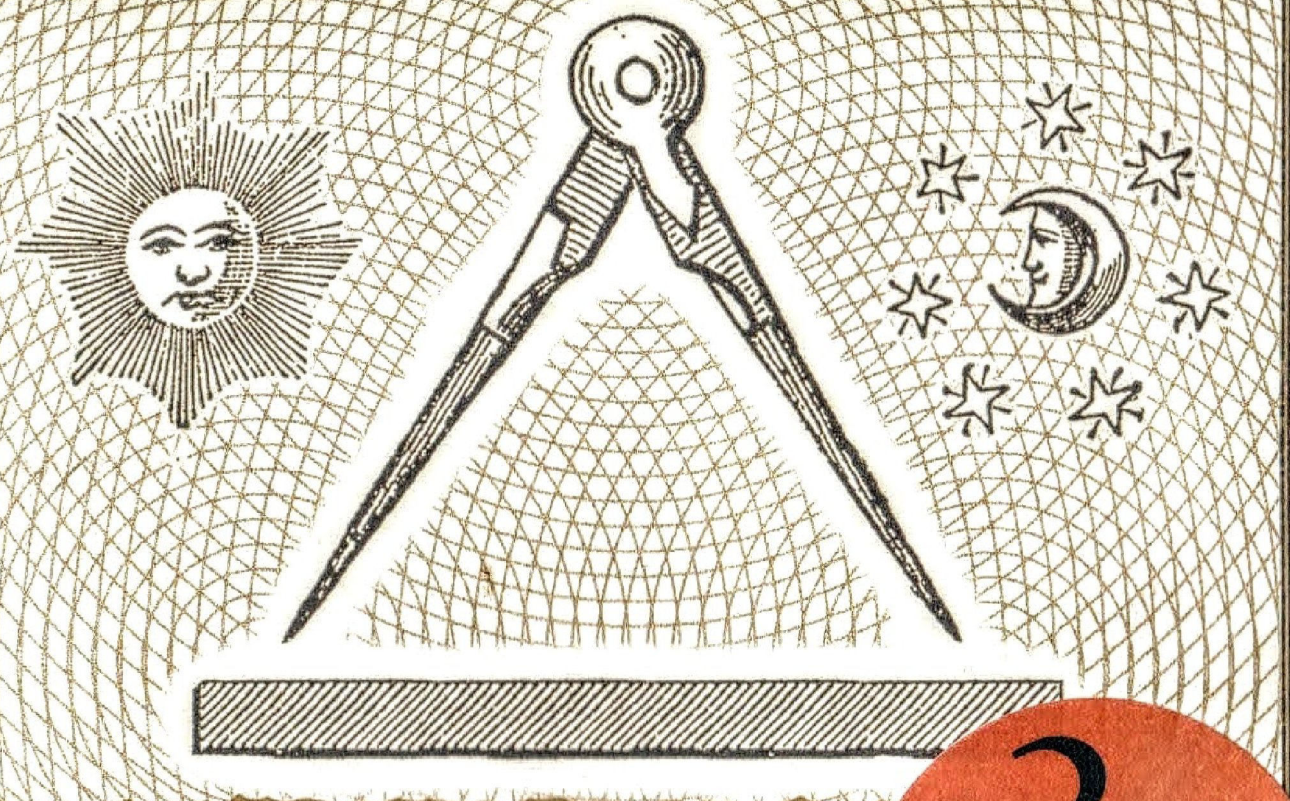
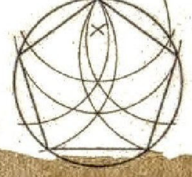
the ultimate timeless pocket guide to

RULER & COMPASS

PRACTICAL GEOMETRIC CONSTRUCTIONS





Andrew Sutt

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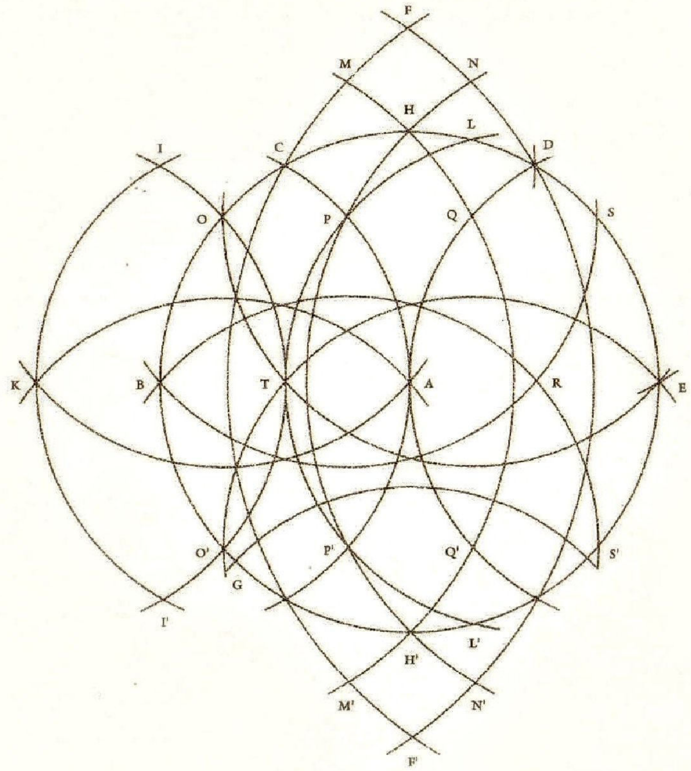

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$PT = \sqrt{2}$	$QQ' = \sqrt{7}$	$BD = \sqrt{12}$	$FK = \sqrt{17}$	$KS = \sqrt{22}$
$DR = \sqrt{3}$	$AF = \sqrt{8}$	$HK = \sqrt{13}$	$KN = \sqrt{18}$	$MM' = \sqrt{23}$
$AB = \sqrt{4}$	$BR = \sqrt{9}$	$BS = \sqrt{14}$	$KD = \sqrt{19}$	$MN' = \sqrt{24}$
$HT = \sqrt{5}$	$BL = \sqrt{10}$	$LL' = \sqrt{15}$	$FG = \sqrt{20}$	$KE = \sqrt{25}$

(after L. Mascheroni & A.N. Kostovskii)

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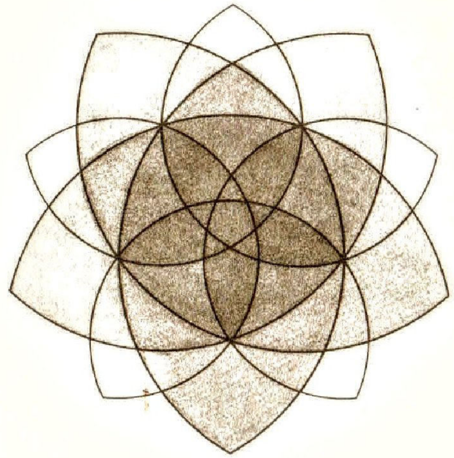
RULER & COMPASS

Practical Geometric Constructions



by

Andrew Sutton

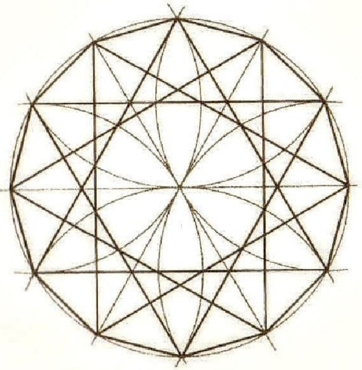
In loving memory of John Michell.

Thanks to Nikki for putting up with another book.
Thanks to Robert and Burkard for checking the constructions and the text.
Thanks too to Mr. M.C. Percival, who first introduced me to ruler and compass constructions.
Special thanks to all those geometers whose work is included in this small book.

The following sources have been essential in this book's compilation:

J.L. Berggren, *Episodes in the Mathematics of Medieval Islam;* Wm. Fitch Cheney, *Can We Outdo Mascheroni?* (article); Robert Dixon, *Mathographics;* T.W. Good, *Plane & Solid Geometry;* Jay Hambidge, *The Elements of Dynamic Symmetry;* Joseph Harrison & G.A. Baxandall, *Practical Plane & Solid Geometry for Advanced Students;* Robin Hartshorne, *Geometry: Euclid and Beyond;* T. L. Heath, *The Thirteen Books of Euclid's Elements;* E.W. Hobson, *Squaring the Circle;* Jay Kappraff, *A Secret of Ancient Geometry* in *Geometry at Work;* A.N. Kostovskii, *Geometrical Constructions Using Compasses Only;* Mark A. Reynolds, *Geometric and Harmonic Means and Progressions* (article); Paul Rosin, *On Serlio's Construction of Ovals* (article); A.S. Smogorzhevskii, *The Ruler in Geometrical Constructions;* Henry J. Spooner, *The Elements of Geometrical Drawing;* Dwarka Nath Yajvan & Prof. G.Thibout, *Baudhayan Sulbasutram;* Jim Loy's geometry web pages, especially on trisection, *www.jimloy.com;* Forum Geometricorum (online journal), *forumgeom.fau.edu*

Apart from a few adaptations and simple constructions by the author, if no particular source for a construction is mentioned it is quite standard and well known from many sources.



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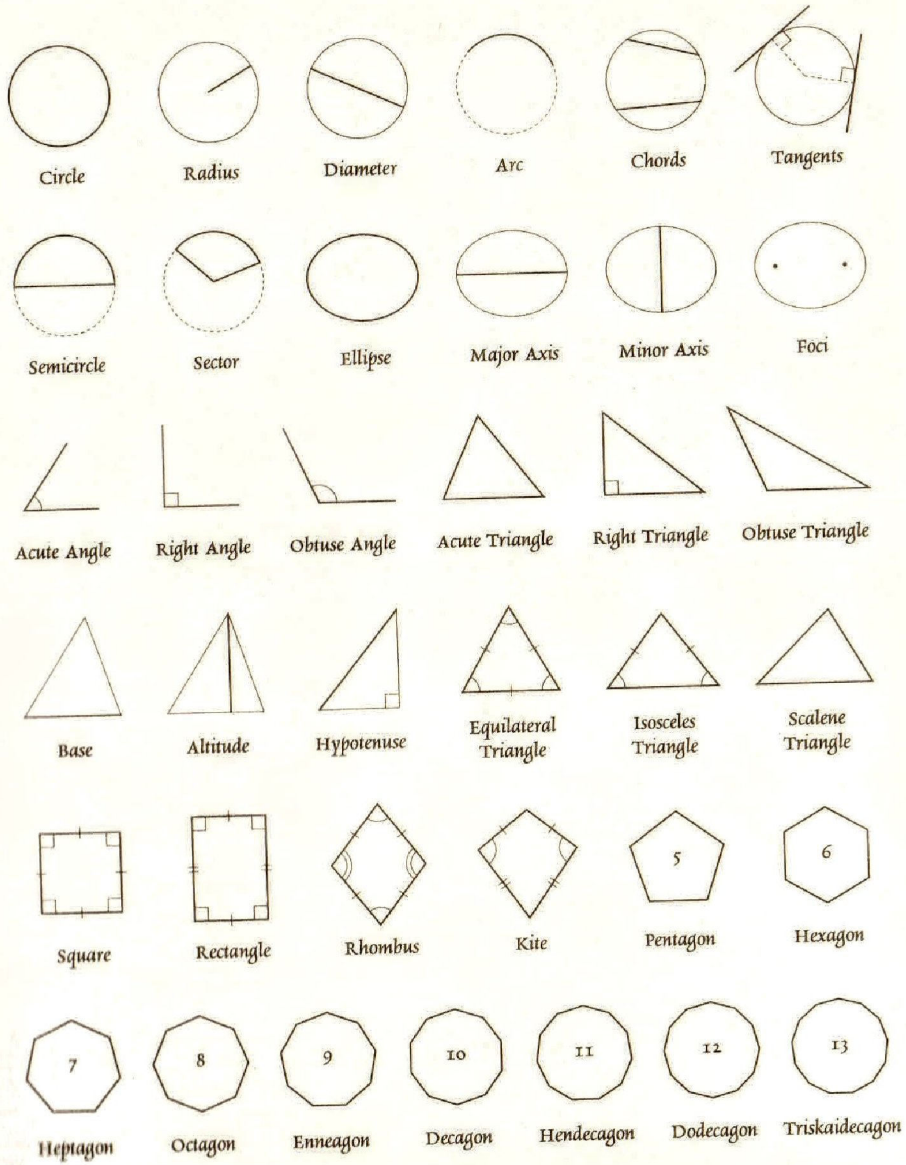


INTRODUCTION

The art of geometric construction can be traced back to the widespread, possibly universal, practice of marking out simple forms and measures on the Earth using pegs and cords – *geometry*, literally Earth measure. Examples include ancient Egyptian rope stretchers, or *harpenodaptai*, who re-established land boundaries after the annual Nile flood, and ancient Indian altar construction techniques found in the Vedic *Śulbasūtras,* the oldest surviving texts with geometric instruction. In time this became the more familiar mathematical discipline, practiced at a smaller scale. Plato (d. *ca.* 347 BC) first stipulated the strict use of only ruler and compass, the ideal simple forms of straight line and circle.

This book is intended as a small practical guide to the field, inspired by the artisans' manuals penned by Abu'l-Wafa' al-Buzjani (d. 998) and Albrecht Dürer (d. 1528). Some mathematical context and history is given, but no proofs. Unless noted, all constructions are mathematically exact. Readers are highly encouraged to try their hand at some of them – there is no substitute for actually taking ruler and compass to paper.

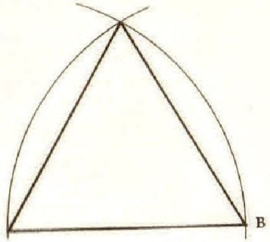
This book uses a simple code. *Line AB* means *draw the straight line that passes through A and B*. *Segment* is used in place of *line* for the section of a straight line defined by two endpoints. *Circle O-A* means *draw a circle centred at O and passing through A*. *Circle radius AB centre O* means *draw a circle of compass opening length AB centred at O*. *Arc* is used in place of *circle* for drawing only part of the circle. Sometimes, extra points are given to help improve accuracy when drawing, for example, *line ACB*, or *circle O-AB*. Newly found points are noted in brackets. Occasionally a line made possible by new points is assumed drawn, and merely noted, and stages may also be grouped together for brevity. Fear not. All will be clear.

FUNDAMENTALS

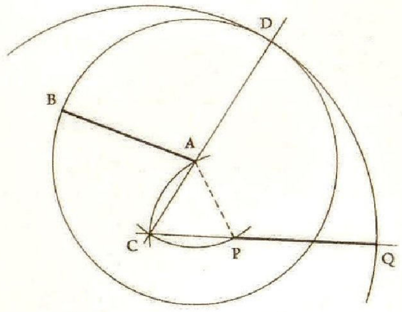
triangles and angle-copying

The Elements of Euclid (*fl.* 300 BC) is one of the greatest works of mathematics ever written. Starting from simple axioms Euclid proceeds logically to prove theorems in geometry and number theory. His axioms include the possibility of marking a line between any two points, and marking a circle centred on one point and passing through another. He never, however, measures a distance between two points and draws a circle of that radius centred elsewhere – for he is using a *collapsible compass.*

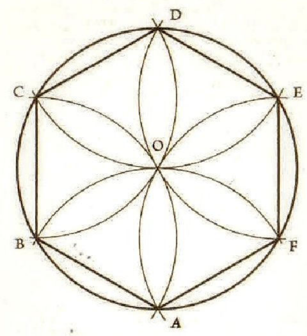
After first constructing an equilateral triangle (construction 1) Euclid proves that his simpler collapsible compass can take a distance between two points and transfer it to draw a circle of that radius centred at any other point (construction 2). A practical ruler and compass construction that uses a compass directly to transfer distance can therefore still be considered Euclidean, the additional steps of construction 2 are simply not drawn.



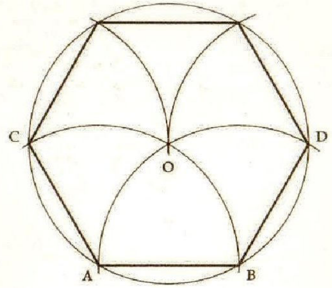
1. Equilateral triangle on a given side:
1. Arc A-B; 2. Arc B-A & complete



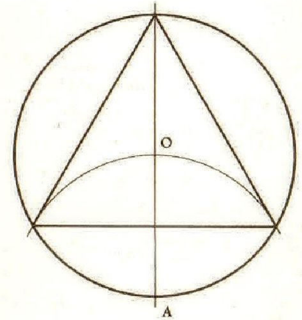
2. Transferring any distance AB:
1. Arcs A-P, P-A (C); 2. Lines CA, CP;
3. Circle A-B (D); Arc C-D (Q);
PQ = AB



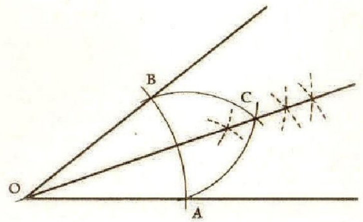
3. Regular hexagon in a circle:
1. Arc A-O (B, F); 2. Arc B-OA (C);
3. Arc C-OB (D); 4. Arc D-OC (E);
5. Arc E-ODF; 6. Arc F-OEA & complete



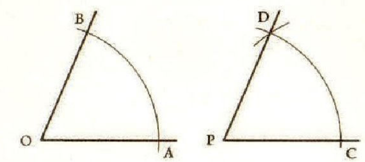
4. Regular hexagon on a given side:
1. Arcs A-B, B-A (O); 2. Circle O-AB
(C, D); 3. Arcs C-O, D-O & complete



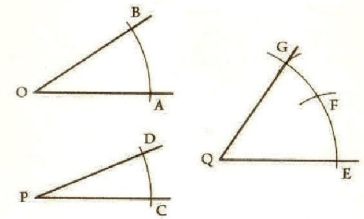
5. Equilateral triangle in a circle:
1. Line through centre O (A);
2. Arc A-O & complete



6. Bisecting an angle:
1. Arc centre O (A, B); 2. Arcs A-B, B-A
(C), alternatives shown dashed; 3. Line OC;
∠AOC = ∠BOC = 1/2 ∠AOB



7. Copying an angle on a given line:
1. Arc centre O (A, B); 2. Arc radius OA
centre P (C); 3. Arc radius AB centre C (D);
4. Line PD; ∠CPD = ∠AOB



8. Adding two angles on a given line:
1. Arc centre O (A, B); 2. Arc radius OA
centre P (C, D); 3. Arc radius OA centre
Q (E); 4. Arc radius AB centre E (F);
5. Arc radius CD centre F (G); 6. Line QG;
∠EQG = ∠AOB + ∠CPD

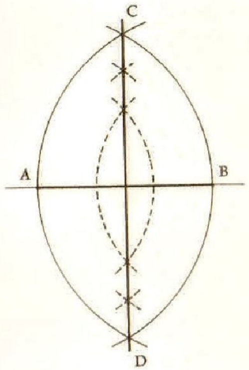
PERPENDICULARS

sticking straight out

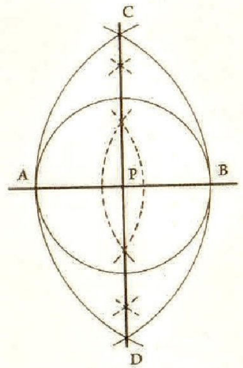
For our purposes a ruler is any unmarked straight edge. If you use a ruler with marked units of length the markings should be ignored. Measurements may only be taken from the construction itself.

A perpendicular bisector intersects a segment at its midpoint (sometimes the construction is used simply to find this midpoint). Note that the arcs constructing a perpendicular bisector do not have to be drawn in full, they simply need to be of equal radius and they must cross. Examples are shown below as dotted lines (constructions 9, 10, and 11).

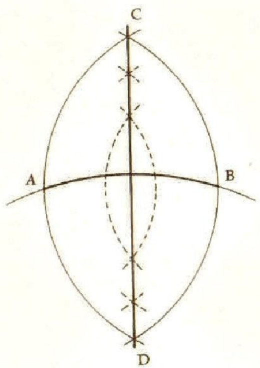
In construction 14 opposite any two arcs suffice, and alternatives are shown dotted. This construction can be used even when the point lies beyond the end of the line, with both arcs turned the same way.



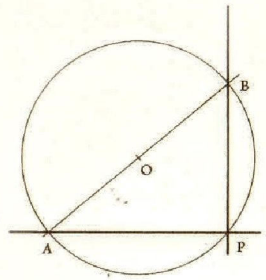
9. Perpendicular bisector on a given segment AB: 1. Arcs of equal radius centres A, B (C, D); 2. Line CD



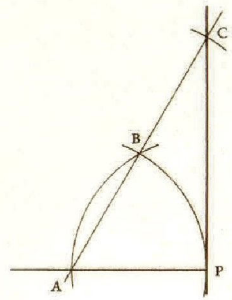
10. Perpendicular through a point P on a line: 1. Circle centre P (A, B); 2. Arcs of equal radius centres A, B (C, D); 3. Line CPD



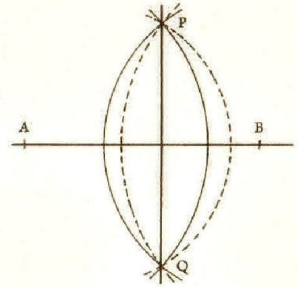
11. Perpendicular bisector on a given arc AB: 1. Arcs of equal radius centres A, B (C, D); 2. Line CD



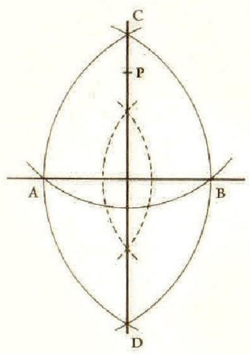
12. Perpendicular through a point P on a line: 1. For any point O not on the line, circle O-P (A); 2. Line AO (B); Line PB is perpendicular to line AP



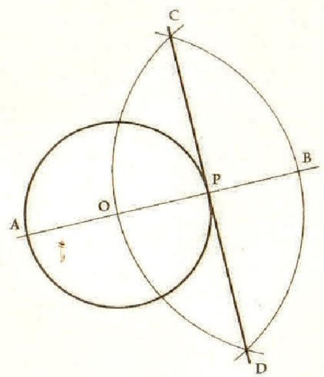
13. Perpendicular at the end of a given segment: 1. Arc any radius, centre P (A); 2. Arc A-P (B); 3. Line AB; 4. Arc B-AP (C); Line PC is perpendicular to line AP



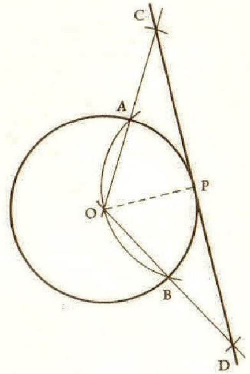
14. Perpendicular through a point P not on a line: 1. For any point A on the line, arc A-P; 2. For any point B on the line, arc B-P (Q); Line PQ is perpendicular to line AB



15. Perpendicular through a point P not on a line: 1. Arc any suitable radius centre P (A, B); 2. Arcs of equal radius centres A, B (C, D); 3. Line CPD



16. Tangent to a circle: 1. Line OP (A); 2. Arc radius PA centre O (B); 3. Arc B-O (line CPD); Line CPD is tangent to circle at P



17. Tangent to a circle: 1. Arc P-O (A, B); 2. Lines OA, OB; 3. Arcs A-O, B-O (CPD); Line CPD is tangent to circle at P

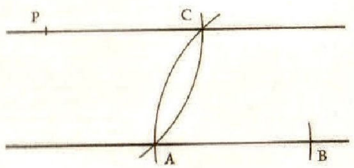
PARALLELS

staying on the level

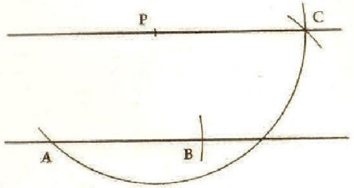
In Euclidean geometry straight lines continue forever in both directions, parallel lines are straight lines in the same plane that never meet, and there is only one parallel to a line through a given point. This is not so for non-Euclidean geometries such as those on spherical or hyperbolic surfaces.

Constructions 18 to 20 for a parallel to a line through a point not on the line are the simplest possible; constructions 18 and 19 use three arcs of arbitrary fixed radius, construction 20 uses two arcs with one change of compass opening. Construction 23 is a bit of a cheat as the parallel is not drawn through two established points; however, it is simple and very reliable. In practical work it can perhaps be considered not dissimilar to transferring a distance with the compass – the result is something possible in Euclidean construction but a short cut is used to draw it.

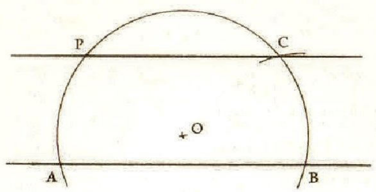
Many of the basics covered so far are used in subsequent constructions and included as instructions, for example *perpendicular to* AB *through* P. It is well worth being familiar with them all before proceeding.



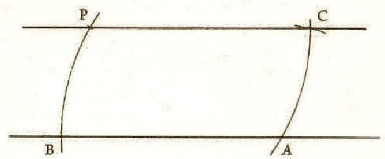
18. Parallel through a given point P:
1. Arc any suitable radius centre P (A);
2. Arc same radius centre A (B);
3. Arc same radius centre B (C);
Line PC is parallel to line AB



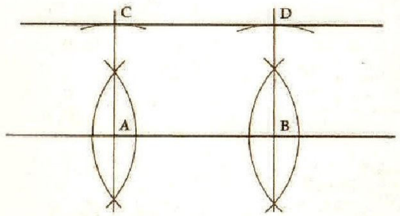
19. Parallel through a given point P:
1. Arc any suitable radius centre P (A);
2. Arc same radius centre A (B);
3. Arc same radius centre B (C);
Line PC is parallel to line AB



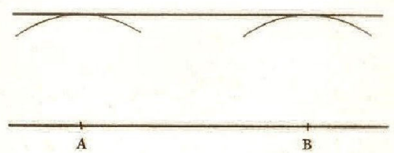
20. Parallel line through a given point P:
1. Arc O-P any suitable centre O (A, B);
2. Arc radius AP centre B (C);
Line PC is parallel to line AB



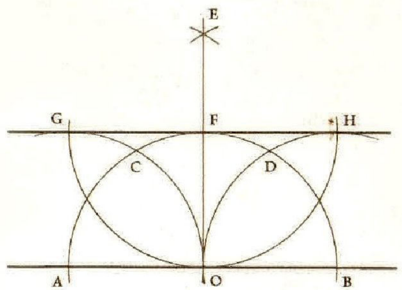
21. Parallel line through a given point P:
1. Arc centre P (A); 2. Arc A-P (B);
3. Arc radius BP centre A (C);
Line PC is parallel to line AB



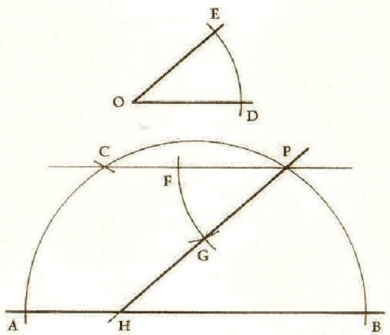
22. Parallel line at a given distance:
1. Any two perpendiculars to line (A, B);
2. Arcs radius equal to given distance, centres A, B (C, D);
Line CD is parallel to AB



23. Parallel line at a given distance:
1. Arcs radius equal to given distance, centres any two points A, B on the line; 2. Line touching arcs as shown is parallel to AB



24. Parallel line at a given distance:
1. Arc radius equal to given distance, centre any point O on the line (A, B); 2. Arcs A-O, B-O (C, D); 3. Arcs C-D, D-C (E);
4. Line EO (F); 5. Arc F-O (G, H);
Line GFH is parallel to AB

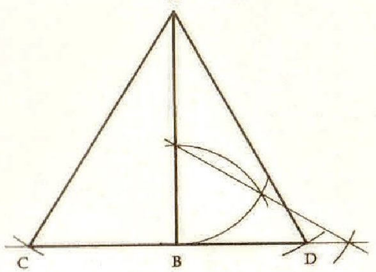


25. Line through a point P not on a line, forming a given angle to the line:
1. Parallel to line AB through P (line CP);
2. Arc centre O (D, E); 3. Arc radius OE centre P (F); 4. Arc radius DE centre F (G); 5. Line GP (H); $\angle BHP = \angle DOE$

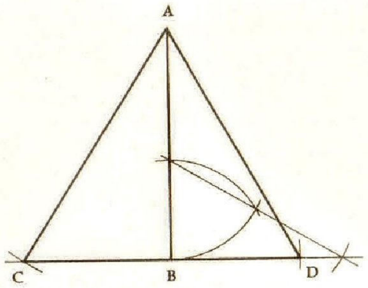
TRIANGLES
three's a polygon

Simple triangular constructions are well worth learning. For example, to construct any triangle given the three side lengths take a segment equal to one length and draw arcs with radii equal to the other two lengths centred on either endpoint; the intersection of these arcs is the triangle's third vertex. Or to construct a right triangle given the two shorter side lengths simply mark off these lengths with arcs centred on the intersection of two perpendicular lines.

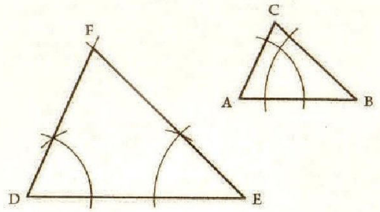
Construction 26 can also be used for an isosceles triangle given its altitude and non-base side. Constructions 30–32 rely on what may be the oldest mathematical theorem, the theorem of Thales of Miletus (d. *ca.* 546 BC): An angle inscribed in a semicircle is a right angle. Construction 30 can be used to construct a rectangle given its diagonal and one side: with the diagonal as hypotenuse, simply complete the construction in both semicircles of a complete circle. Construction 33 is from *The Elements*.



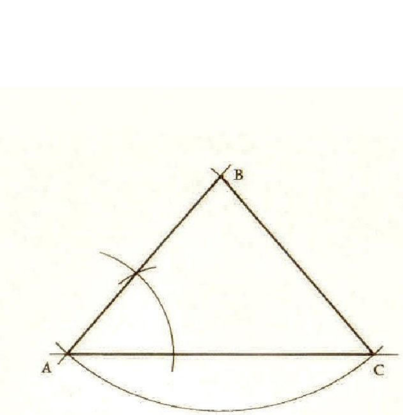
26. Triangle given altitude and two sides:
1. Perpendicular to altitude AB at B;
2. Arc radius one given side, centre A (C);
3. Arc radius other given side, centre B (D)



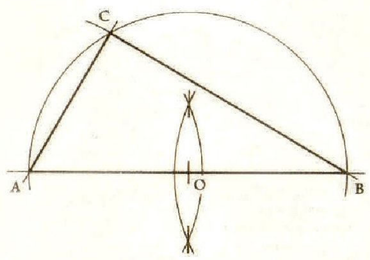
27. Triangle given altitude, base and a side:
1. Perpendicular to altitude AB at B;
2. Arc radius one given side, centre A (C);
3. Arc radius base, centre C (D)



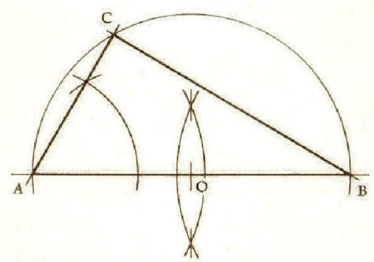
28. Similar triangle on a given base:
1. Make angle at D equal to angle at A;
2. Make angle at E equal to angle at B (F)



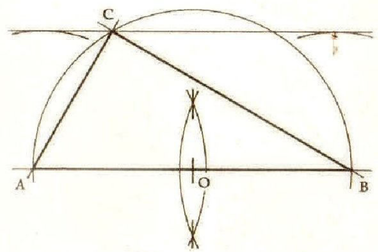
29. Isosceles triangle given side and base angle:
1. Make angle at A equal to given base angle;
2. Arc radius equal to given side, centre A (B);
3. Arc B-A (C)



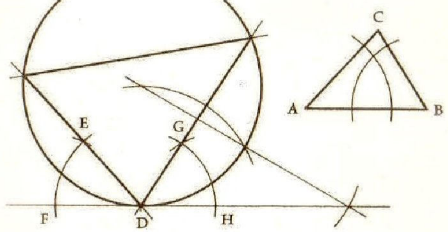
30. Right triangle given hypotenuse & side:
1. Segment AB equal to given hypotenuse;
2. Midpoint of AB (O); 3. Arc O-AB;
4. Arc radius of given side, centre A (C)



31. Right triangle given hypotenuse & angle:
1. Segment AB equal to given hypotenuse;
2. Midpoint of AB (O); 3. Arc O-AB;
4. Make ∠CAB equal to given angle (C)



32. Right triangle given hypotenuse & altitude:
1. Segment AB equal to given hypotenuse;
2. Midpoint of AB (O); 3. Arc O-AB;
4. Parallel to AB at given altitude (C)



33. Similar triangle in a circle:
1. Tangent to circle at D;
2. Make ∠EDF = ∠CAB;
3. Make ∠GDH = ∠CBA & complete

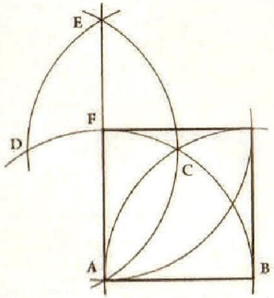
SQUARES & RHOMBUSES

from lines and circles

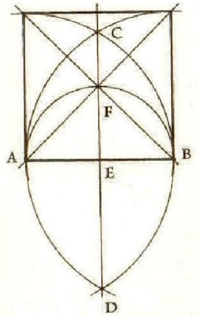
Basic tetragonal constructions are easily mastered. To draw a circle about a square or rectangle first add the diagonals then draw the circle centred on their intersection and passing through the four vertices. To inscribe a circle within a square add the diagonals, find the midpoint of one side, then draw the circle centred on the intersection of the diagonals and passing through the edge midpoint.

The diagonals of squares and rhombuses bisect each other at right angles. The exploration of the relationship of a square's diagonal to its side, $\sqrt{2}:1$, led to the proof that some lengths are incommensurable, i.e., they cannot both be measured by the same unit, no matter how small.

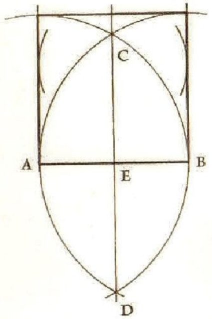
Construction 39 is given in the *Baudhyāna Śulbasūtra* (*ca.* 800–600 BC) as a method for marking out a square altar on an East-West line, known as the *prācī*. It is particularly elegant drawn with full circles.



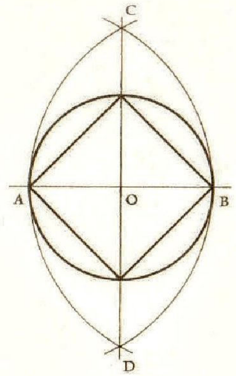
34. *Square on a given side:* 1. *Arcs* A-B, B-A (C); 2. *Arc* C-A (D); 3. *Arc* D-AC (*line* EFA); 4. *Arc* F-A & *complete*



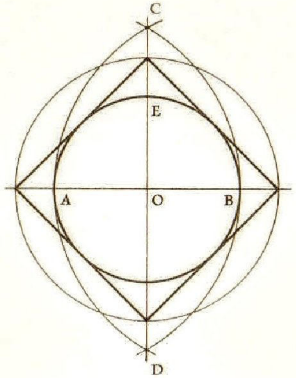
35. *Square on a given side:* 1. *Arcs* A-B, B-A (*Line* CED); 2. *Arc* E-AB (F); 3. *Lines* AF, BF & *complete*



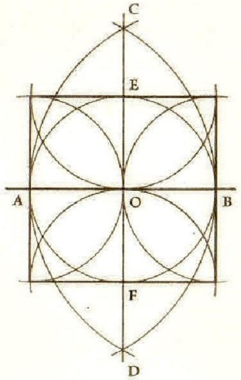
36. *Square on a given side:* 1. *Arcs* A-B, B-A (*Line* CED); 2. *Parallels to* CED *through* A, B & *complete*



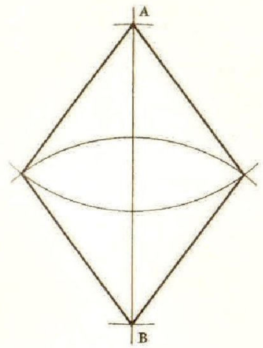
37. *Square in a circle:* 1. *Line through centre* O (A, B); 2. *Arcs* A-B, B-A; 3. *Line* CD & *complete*



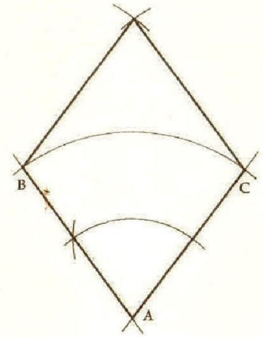
38. *Square around a circle:* 1. *Line through centre* O (A, B); 2. *Arcs* A-B, B-A (*line* CED); 3. *Circle radius* AE *centre* O & *complete*



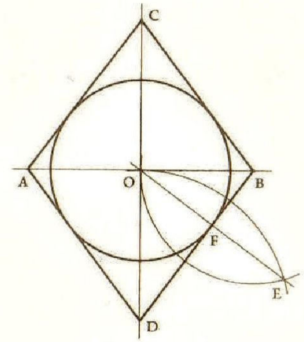
39. *Square set orthogonally on a given line:* 1. *Circle centre* O *on the line* (A, B); 2. *Arcs* A-B, B-A (*line* CEFD); 3. *Arcs* A-O, B-O, E-O, F-O & *complete*



40. *Rhombus given one diagonal and side length:* 1. *Mark diagonal length on a line* (A, B); 2. *Arcs radius of given side length, centres* A, B & *complete*



41. *Rhombus given one angle and side length:* 1. *Copy given angle on a line at* A; 2. *Arc radius of given side length centre* A (B, C); 3. *Arcs* B-A, C-A & *complete*



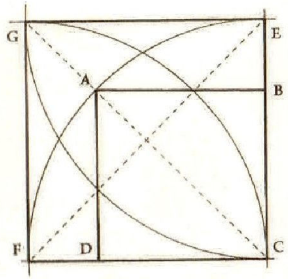
42. *Circle in a rhombus:* 1. *Lines* AB, CD (O); 2. *Arcs* B-O, D-O (*line* OFE); 3. *Circle* O-F

SQUARE AREAS

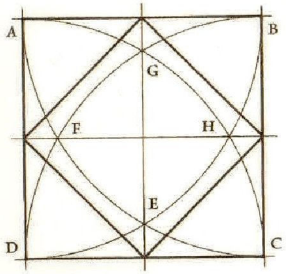
easier than volumes

Around 430 BC the oracle at Delos challenged the Athenians to double the volume of Apollo's cubic altar to avert a plague (*see page 38*). Doubling altars, often squares, by area rather than volume is a recurring theme in the *Śulbasūtras,* as is the addition and subtraction of square areas. Constructions 45 and 46 are from the *Baudhyāna Śulbasūtra,* and both are clear applications of the Pythagorean theorem which states that in a right triangle, the square on the hypotenuse is equal to the sum of the squares on the other two sides. To use construction 43 to halve a square include the dotted diagonals. Construction 48 is based on Abu'l-Wafa's square dissections. The dotted circle shows an alternative way of finding E to H.

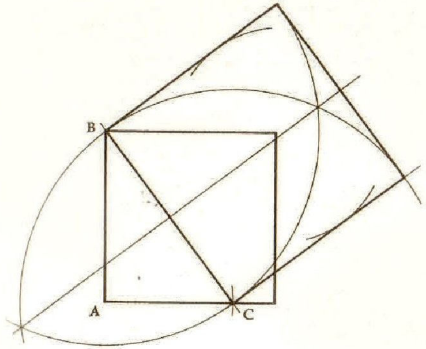
It may be useful to add a drawing tip here. The most common source of error is in the positioning of the compass point, so to reduce this hold the compass at the top with one hand and use the other hand to guide the point in to place – do not swap hands to draw the circle.



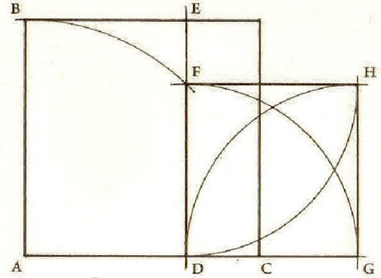
43. *Doubling a square:*
1. *Extend* CB, CD; 2. *Arcs* C-A (E, F);
3. *Arcs* F-C, E-C (G) & *complete;*
To halve square GECF *include dotted lines*



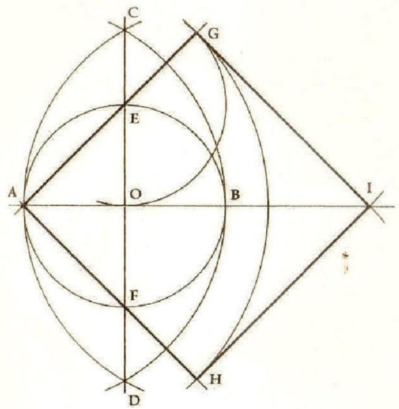
44. *Halving a square:*
1. *Arcs* A-BD, B-AC, C-BD, D-AC (E *to* H); 2. *Lines* EG, FH & *complete*



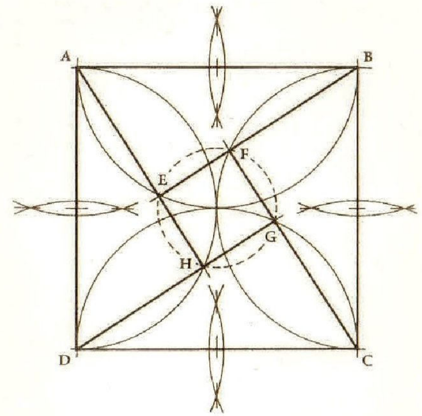
45. *Sum of two squares:*
1. *On the larger square, arc radius equal to side of smaller square, centre* A (C);
2. *Construct square on line* BC *to complete*



46. *Difference of two squares:*
1. *On larger square, arcs radius equal to side of smaller square, centres* A, B (*line* DE);
2. *Extend line* AC; 3. *Arc* A-B (F); 4. *Arc* D-F (G); 5. *Arcs* F-D, G-D (H) & *complete*



47. *Square given difference of side & diagonal:*
1. *Circle on a line, radius of given length, centre* O (A, B); 2. *Arcs* A-B, B-A (*line* CEFD); 3. *Lines* AE, AF;
4. *Arc* E-O (G); 5. *Arc* A-G (H);
6. *Arcs* H-A, G-A (I) & *complete*



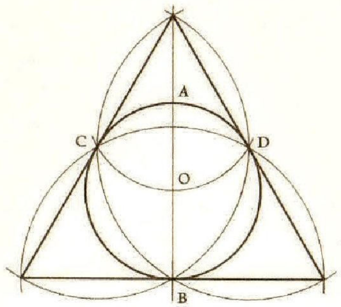
48. *Square & four right triangles in a square:*
1. *Find midpoints of sides;* 2. *Four semicircles through* A *to* D, *centred on midpoints;* 3. *Arcs radius of one side of right triangle, centres* A *to* D (E *to* H);
4. *Lines* AE, BF, CG, DH

HEXAGONS & DODECAGONS

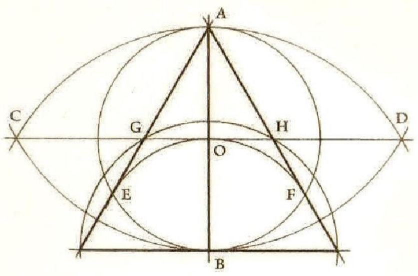
a story of threes and fours

The journey from triangle and square to the 12-sided dodecagon is simple and edifying. Construction 51 can also be used to construct a regular hexagon of given width. The dotted lines shown in construction 54 can be used either as an alternative method, or to complement the arcs – if the arcs and dotted lines all coincide then your drawing is very accurate.

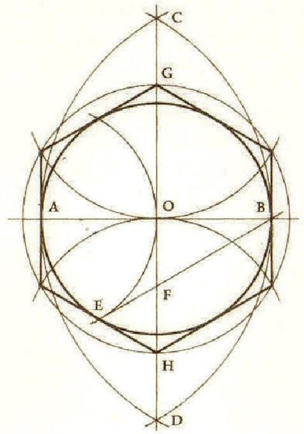
The vertices of a regular polygon are equally spaced points on the circumference of a circle. Except in the case of the equilateral triangle it is always more accurate to draw a regular polygon starting with this circle than with a side, as constructions based on the latter become increasingly unreliable the greater the number of sides and are best used only when a larger more complex design requires this approach. When drawing a regular polygon on a blank page it is best first to draw a central line and then the circle centred on this line.



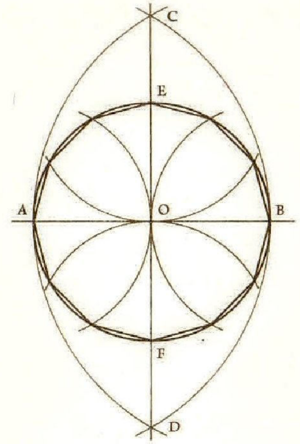
49. *Equilateral triangle around a circle:*
1. *Line through centre* O (A, B); 2. *Arc* A-O (C, D); 3. *Arcs* C-BD, D-BC, B-CD *& complete*



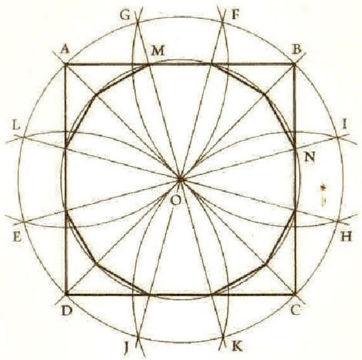
50. *Equilateral triangle on a given altitude:*
1. *Arcs* A-B, B-A (C, D); 2. *Line* CD (O); 3. *Circle* O-AB; 4. *Arc* B-O (E, F); 5. *Lines* AE, AF (G, H); 6. *Arc* B-GH *& complete*



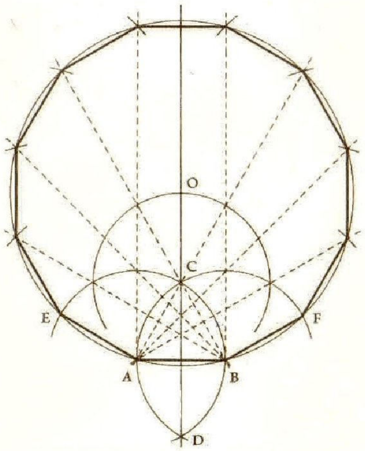
51. *Regular hexagon around a circle:*
1. *Line through centre* O (A, B); 2. *Arcs* A-B, B-A (*line* CD); 3. *Arc* A-O (E); 4. *Line* EB (F); 5. *Circle radius* BF *centre* O (G, H); 6. *Arcs* G-O, H-O *& complete*



52. *Regular dodecagon in a circle:*
1. *Line through centre* O (A, B); 2. *Arcs* A-B, B-A (*line* CEFD); 3. *Arcs* A-O, E-O, B-O, F-O *& complete*



53. *Regular dodecagon in a square:*
1. *Lines* AC, BD (O); 2. *Circle* O-ABCD; 3. *Arcs* A-O, B-O, C-O, D-O (E *to* L); 4. *Lines* EI, LH, GK, FJ (M, N); 5. *Circle* O-MN *& complete*



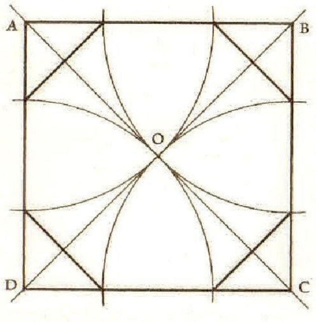
54. *Regular dodecagon on a given side:*
1. *Arcs* A-B, B-A (*line* CD); 2. *Arc radius* AB *centre* C (O); 3. *Circle* O-AB (E, F); 4. *Arcs radius* OA *centres* A, B, E, F *and again on new intersections & complete*

Octagons

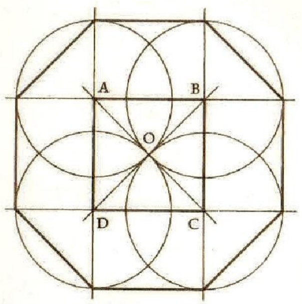
something perfect

The octagon appears naturally from the square. Construction 56 can be combined with constructions 38 or 39 to give an octagon of given side length, starting with half this length as radius for the first circle. Construction 58 instructs the reader to walk a measure around the circumference, meaning that the compass should be opened to the specified measure, and with the same opening further arcs should be drawn round the circumference centred on the new intersections until all the polygon's vertices are found.

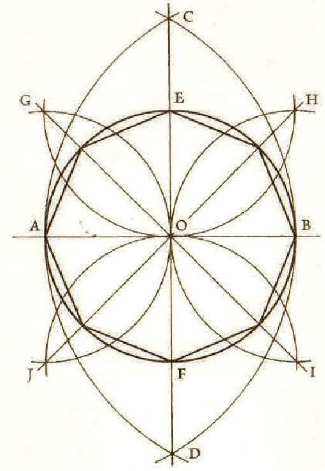
The diagonals of a polygon are the internal lines between its vertices. Take any regular polygon inscribed in a circle of radius 1, or *unit circle*. Multiply together the lengths of the diagonals and two sides which meet at any one vertex and the result will be the number of sides in the polygon. Check it with a square, hexagon, or octagon.



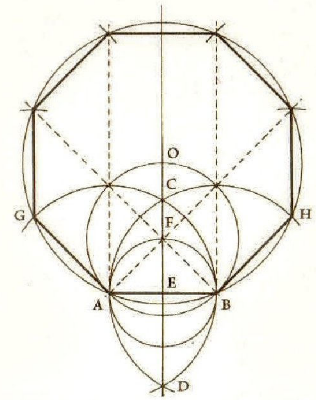
55. *Regular octagon in a square:* 1. *Lines* AC, BD (O); 2. *Arcs* A-O, B-O, C-O, D-O *& complete*



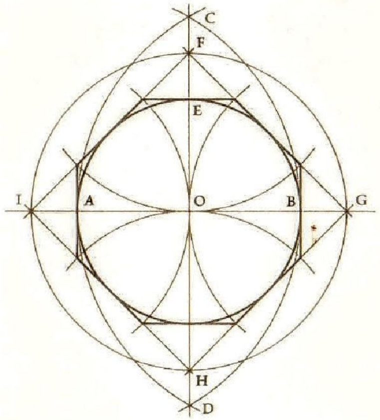
56. *Regular octagon from a square:* 1. *Lines* AC, BD (O); 2. *Circles* A-O, B-O, C-O, D-O; 3. *Lines* AB, BC, CD, DA *& complete*



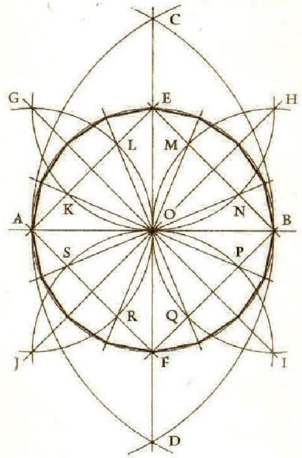
57. *Regular octagon in a circle:* 1. *Line through centre* O (A, B); 2. *Arcs* A-B, B-A (*line* CEFD); 3. *Arcs* A-O, E-O, B-O, F-O (G, H, I, J); 4. *Lines* GI, HJ *& complete*



58. *Regular octagon on a given side:* 1. *Arcs* A-B, B-A (*line* CED); 2. *Circle* E-AB (F); 3. *Circle* F-AB (O); 4. *Circle* O-AB (G, H); 5. *Walk* AB *around circumference from* G, H *& complete*



59. *Regular octagon around a circle:* 1. *Line through centre* O (A, B); 2. *Arcs* A-B, B-A (*line* CED); 3. *Circle radius* AE *centre* O (F *to* I); 4. *Lines* FG, GH, HI, IF; 5. *Arcs* F-O, G-O, H-O, I-O *& complete*



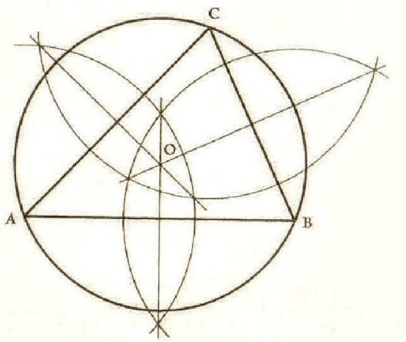
60. *Regular 16-gon in a circle:* 1. *Line through centre* O (A, B); 2. *Arcs* A-B, B-A (*line* CEFD); 3. *Arcs* A-O, E-O, B-O, F-O (G, H, I, J); 4. *Lines* GI, HJ; 5. *Lines* AE, EB, BF, FA (K *to* N, P *to* S); 6. *Lines* KP, LQ, MR, NS *& complete*

TRIANGLE CENTRES

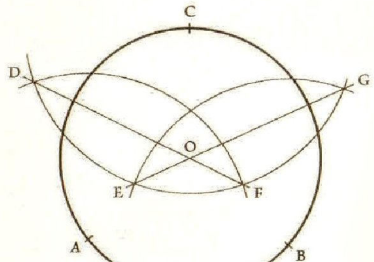
the one in the three

Triangles have hundreds of different centres, but only four of these were known to the ancient Greeks: The *circumcentre* lies at the intersection of the sides' perpendicular bisectors and is the centre of the triangle's circumcircle (this can be used to find the centre of a circle.) The *incentre*, or centre of the incircle, lies at the intersection of the angle bisectors, and can be used to nest circles in regular polygons. The *centroid* lies at the intersection of the medians, lines joining vertices to opposite side midpoints. The *orthocentre* lies at the intersection of the altitudes.

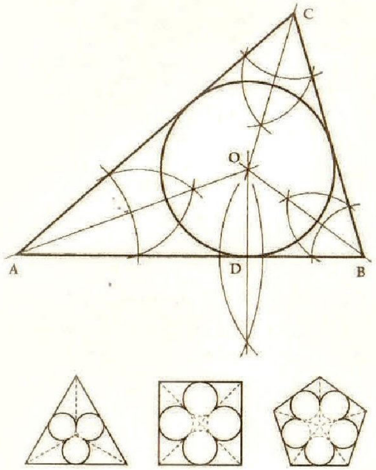
In an equilateral triangle these four centres are concurrent. In all other triangles the circumcentre, centroid, and orthocentre all lie on a line named the Euler line after Leonard Euler, who discovered it in 1765. Also interesting is the fact that the distance between centroid and orthocentre is always twice that between circumcentre and centroid.



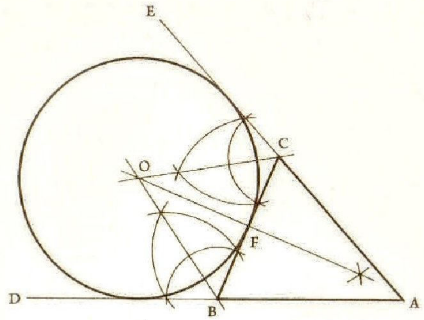
61. *Circumcentre & circumcircle of a triangle:*
1. *Perp. bisectors on* AB, BC, CA (O);
2. *Circle* O-ABC



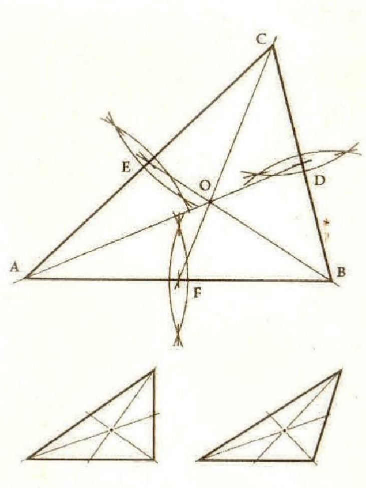
62. *Finding the centre of a circle:*
1. *Choose any three points roughly equally spaced on the circumference* (A, B, C);
2. *Arcs of equal radius centres* A, B, C (D *to* G); 3. *Lines* DF, EG (O)



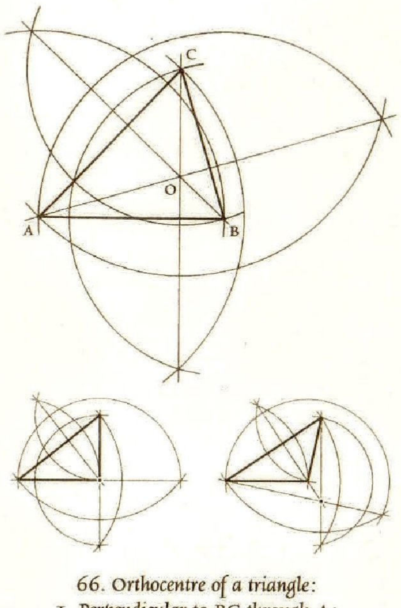
63. *Incentre & incircle of a triangle:*
1. *Bisect* ∠CAB, ∠ABC, ∠BCA (O);
2. *Perp. to* BC *through* O (D); 3. *Circle* O-D



64. *Excircle of a triangle:*
1. *Extend sides* AB *and* AC (D, E); 2. *Bisect* ∠BCE, ∠CBD (O); 3. *Perpendicular to* BC *through* O (F); 4. *Circle* O-F



65. *Centroid of a triangle:*
1. *Find midpoints of* BC, CA, AB (D, E, F);
2. *Lines* AD, BE, CF (O)



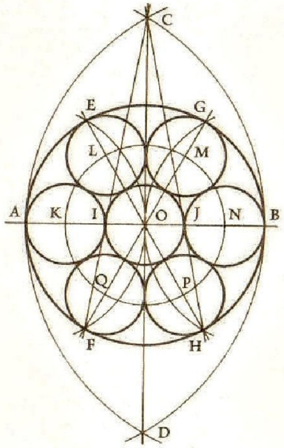
66. *Orthocentre of a triangle:*
1. *Perpendicular to* BC *through* A;
2. *Perpendicular to* AB *through* C;
3. *Perpendicular to* AC *through* B (O)

Inscribed Circles

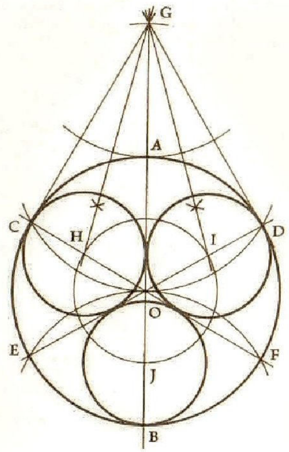
and semicircles

Groups of tangent circles can be very unforgiving to construct. The eye seems particularly sensitive to the meeting of two circles and if they fall short or overlap, even slightly, the whole idea can appear betrayed by poor drawing. To help prevent this it can be useful at some stages to take compass openings from several lengths that should be equal and judge a careful average by eye. This is shown in the instructions using the equals sign for the lengths to be used, e.g., in constructions 67 and 68.

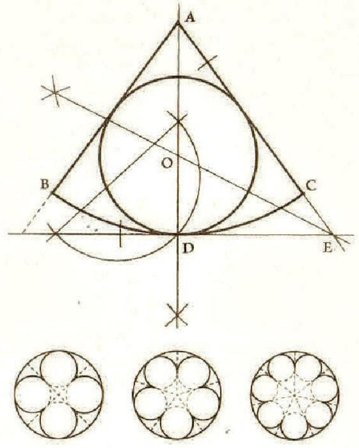
As with construction 63 previously, constructions 69–72 demonstrate principles useful in tracery-style nested circle and semicircle design.



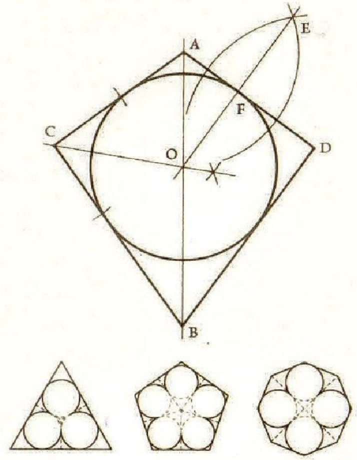
67. *Seven circles in a circle:*
1. *Line through centre* O (A, B); 2. *Arcs* A-B, B-A (*line* CD); 3. *Arcs* A-O, B-O (*lines* EH, GF); 5. *Lines* CF, CH (I, J); 6. *Circle radius* AI = IJ = JB, *centre* O (K *to* Q); 7. *Circles radius* OI = OJ *centres* K *to* Q



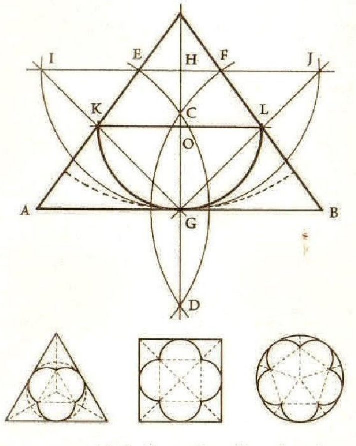
68. *Three circles in a circle:*
1. *Line through centre* O (A, B); 2. *Arcs* A-O, B-O (*lines* CF, DE); 3. *Arc radius* AB *centre* O (G); 5. *Bisect* ∠OGC, ∠OGD (H, I); 6. Circle O-HI (J); 7. *Circles radius* HC = ID = JB *centres* H, I, J



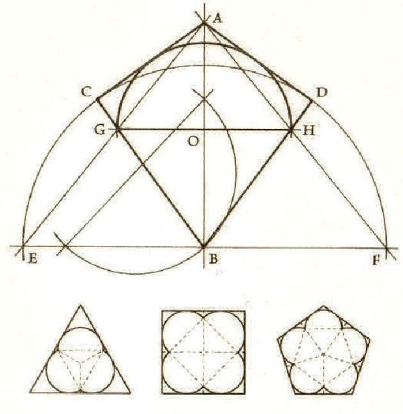
69. *Circle in a sector:*
1. *Bisect* ∠BAC (D); 2. *Perpendicular to* AD *on* D; 3. *Extend side* AC (E); 4. *Bisect* ∠AED (O); 5. *Circle* O-D



70. *Circle in a kite:*
1. *Line* AB; 2. *Bisect* ∠ACB (O); 3. *Arcs* A-O, D-O (E); 4. *Line* OE (F); 5. *Circle* O-F



71. *Semicircle in an isosceles triangle:*
1. *Arcs centres* A, B (C *to* F); 2. *Line* CD (G); 3. *Line* EF (H); 4. *Arc* G-H (I, J); 5. *Lines* IG, JG (K, L); 6. *Line* KL (O); 7. *Arc* O-KGL



72. *Semicircle in a kite that has two right angles:*
1. *Line* AB; 2. *Perpendicular to* AB *on* B; 3. *Arc* B-CD (E, F); 4. *Lines* AE, AF (G, H); 5. *Line* GH (O); 6. *Arc* O-GH

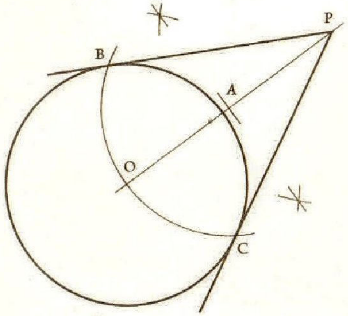
TANGENTS

just a touch

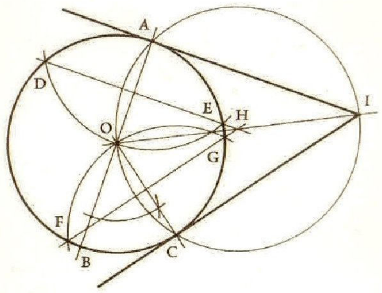
To draw a tangent, a line that just touches a circle, simply put ruler to circumference and trace a line. This is not a Euclidean construction but will suffice if done carefully. Usually, however, and in most practical applications, the point of contact needs to be exact, and, since this cannot be gauged well by eye, it is generally preferable to construct tangents with more rigor. Various techniques for doing this are shown here.

Construction 77 works for separate, touching or overlapping circles, but not for those of equal radius, and poorly when the circles are of a similar size. If two circles touch, their internal tangent is the tangent at their point of contact, if they overlap there are no internal tangents.

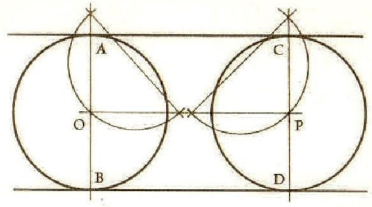
Drawing lines tangent to circles is quite forgiving, drawing circles simultaneously tangent to more than one line is not – construction 79 is actually the hardest construction on these pages to get just right.



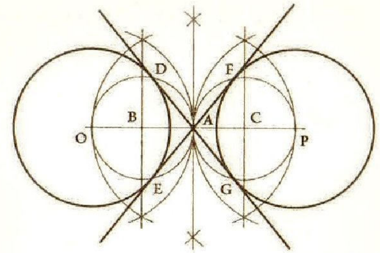
73. Tangents to a circle from a point P:
1. Find midpoint of OP (A); 2. Arc A-O (B, C); 3. Lines BP, CP



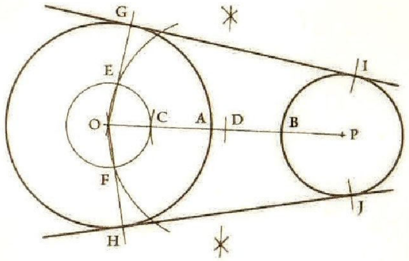
74. Tangents to a circle forming a given angle:
1. Line through centre O (A, B); 2. Make given angle on OB at O (C); 3. Arcs A-O, C-O (D to G); 4. Lines DE, FG (H); 5. Line OH; 6. Circle H-O (I); 7. Lines IA, IC



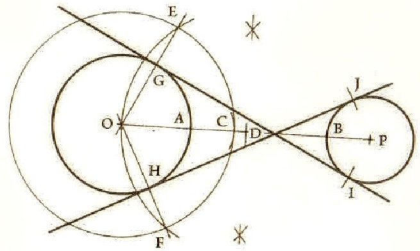
75. External tangent to two equal circles:
1. Line OP; 2. Perpendiculars to OP at O,P (A to D); 3. Lines AB, CD



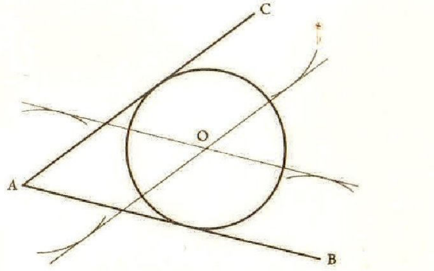
76. Internal tangents to two equal circles:
1. Find midpoint of OP (A); 2. Find midpoints of OA, PA (B, C); 3. Circles B-O, C-P (D to G); 4. Lines DG, EF



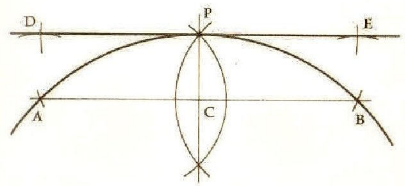
77. External tangents to two unequal circles:
1. Line OP (A, B); 2. Arc radius PB centre A (C); 3. Circle O-C; 4. Find midpoint of OP (D); 5. Arc D-O (E, F); 6. Lines OE, OF (G, H); 7. Arcs radius PE = PF centres G, H (I, J); 8. Lines GI, HJ



78. Internal tangents to unequal circles:
1. Line OP (A, B); 2. Arc radius PB centre A (C); 3. Cirlce O-C; 4. Find midpoint of OP (D); 5. Arc D-O (E, F); 6. Lines OE, OF (G, H); 7. Arcs radius PE = PF centres G, H (I, J); 8. Lines GI, HJ



79. Circle of given radius tangent to two lines:
1. Parallel lines to AB, AC at distance equal to given radius (O);
2. Circle same radius centre O

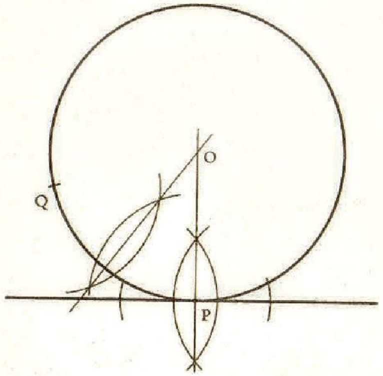


80. Tangent to an arc through given point P:
1. Arc any radius centre P (A, B); 2. Line AB; 3. Perpendicular to AB through P (C); 4. Arcs radius PC centres A, B; 5. Arcs radius AC = BC centre P (line DPE)

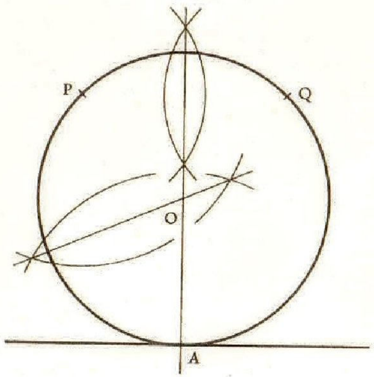
MORE TANGENTS

will it ever end

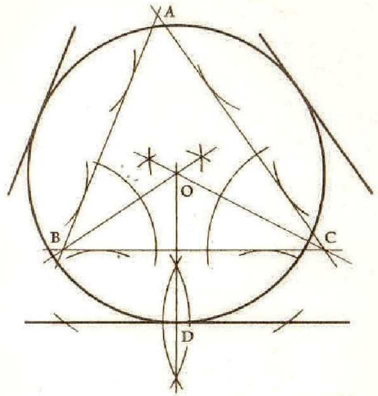
The compass is a mechanical embodiment of the concept of a circle: an opening is set, one point fixed, and the other traces all the equidistant points from that centre. The ruler, however, rather than being a mechanical device embodying the concept of a straight line that draws the shortest distance between two points, is a model of a straight line, along which we trace a copy. Compare this to ancient peg and cord geometry, where the tension in the cord produces the straight line and the same tension keeps points equidistant to trace a circle. Mechanical devices do exist for making straight lines but they are not practical for drawing. To check if your ruler is straight draw a line through two points a good distance apart then turn it half a full turn and, using the same edge, draw another line. If the two lines enclose any space at all then your ruler is not straight.



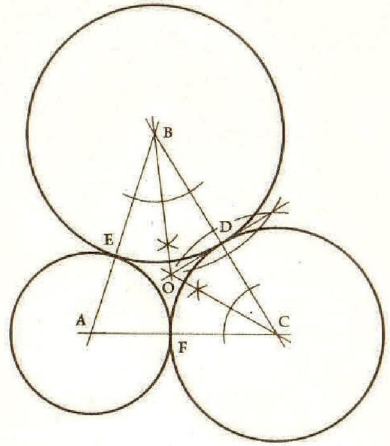
81. *Circle tangent to a line through points* P *(on the line)* & Q *(not on the line):* 1. *Perpendicular to line through* P; 2. *Perp. bisector on* PQ (O); 3. *Circle* O-PQ



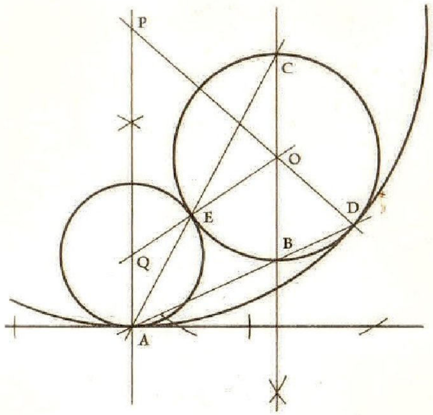
82. *Circle tangent to a line through two points* P & Q *equidistant from the line:* 1. *Perp. bisector on* PQ (A); 2. *Perp. bisector on* AP (O); 3. *Circle* O-APQ



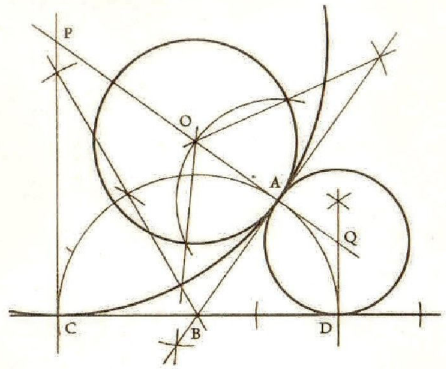
83. *Circle touching three segments:* 1. *Parallels to all three segments at equal distances* (A, B, C); 2. *Bisect* ∠ABC, ∠ACB (O); 3. *Perpendicular to one segment through* O (D); 4. *Circle* O-D



84. *Three touching circles given their centres:* 1. *Lines* AB, BC, CA; 2. *Bisect* ∠ABC, ∠ACB (O); 3. *Perpendicular to* BC *through* O (D); 4. *Circles* B-D, C-D (E, F); 5. *Circle* A-EF



85. *Circles tangent to a circle, centre* O, *and a line at a given point* A: 1. *Perpendiculars to line through* A, O (B, C); 2. *Lines* AB, AC (D, E); 3. *Lines* OD, OE (P, Q); 4. *Circles* P-A, Q-A



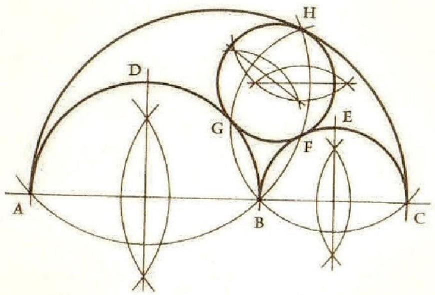
86. *Circles tangent to a line and a circle, centre* O, *at a given point* A: 1. *Line* OA; 2. *Tangent at* A (B); 3. *Arc* B-A (C, D); 4. *Perpendiculars to line at* C, D (P, Q); 5. *Circles* P-C, Q-D

MORE INSCRIBED CIRCLES

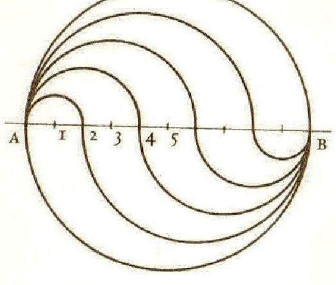
circles within circles

Inscribed circles are widely used in medieval tracery, but it is likely that the original designs were made using simple rules of thumb, along with trial and error, rather than accurate techniques like those shown here.

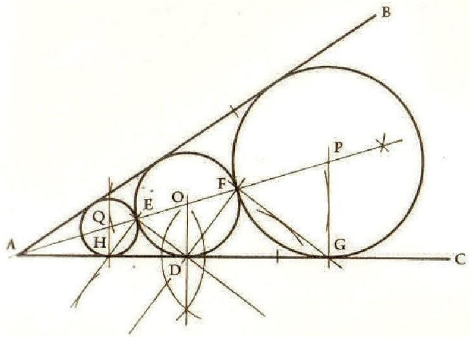
Construction 87 is by Leon Bankoff (d. 1997), and shows an *arbelos,* the ancient Greek word for a shoemaker's knife, and also the shape formed by three semicircles on the same line, each touching the others. Construction 88 is particularly appealing, as the areas and perimeters of all sections are equal, for any number of divisions. The inscribed circles in construction 90 have radii in the proportion 3 : 2 : 1. Constructions 89 and 91 are more involved but apply to any situation fitting their starting points. For example, to inscribe 22 circles perfectly in a regular heptagon join its vertices to its centre, use construction 91 in each of the isosceles triangles formed to find 21 circles, and finally add a central one.



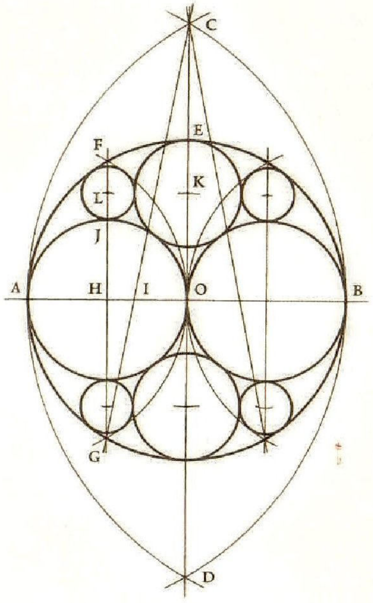
87. *Circle in an arbelos:*
1. *Perp. bisector on* AB (D);
2. *Perp. bisector on* BC (E);
3. *Arcs* D-AB, E-CB (F, G, H);
4. *Circle through* F, G, H *to complete*



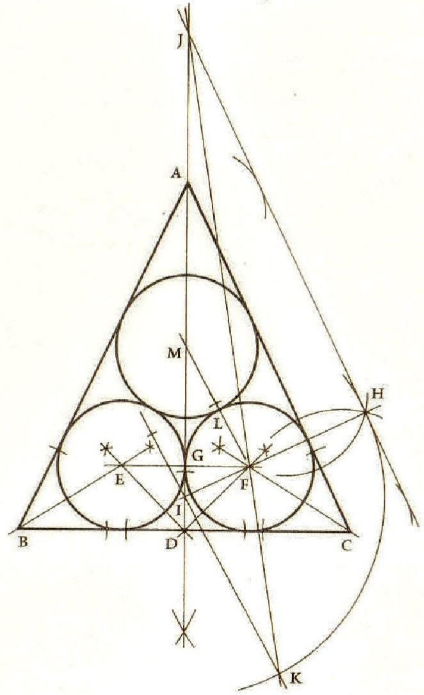
88. *Equal division of a circle:*
1. *Divide a line* AB *into any even number of equal parts (in this case ten)*;
2. *Semicircles centres* 1, 2, 3... *through* A;
3. *Repeat on other side through* B *as shown*



89. *Tangent circles between two lines:*
1. *Bisect* ∠BAC; 2. *Perpendicular to* AB *through any point* O *on bisector* (D);
3. *Circle* O-D (E, F); 4. *Lines* ED, FD;
5. *Parallel to* ED *through* F (G);
6. *Parallel to* FD *through* E (H);
7. *Parallels to* OD *through* G, H (P, Q);
8. *Circles* P-G, Q-H;
Repeat as needed for more circles



90. *Eight circles in a circle:*
1. *Line through centre* O (A, B);
2. *Arcs* A-B, B-A (*line* CED);
The following steps should be mirrored horizontally and vertically as needed;
3. *Arc* A-O (*line* FHG); 4. *Line* CG (I);
5. *Circle* H-O (J); 6. *Arc radius* IO *centre* E (K);
7. *Circle* K-E; 8. *Arc radius* HI *centre* J (L);
9. *Circle* L-J



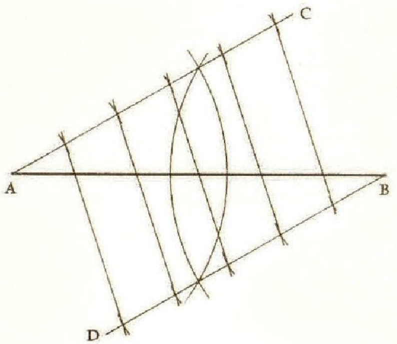
91. *Three circles in an isosceles triangle:*
1. *Perpendicular on* BC (D); 2. *Bisect* ∠ABC, ∠ACB; 3. *Bisect* ∠ADB, ∠ADC (E, F);
4. *Line* EF (G); 5. *Circles* E-G, F-G; 6. *Arcs centres on* AC, *through* F (*line* HI); 7. *Parallel to* AC *through* H (J); 8. *Arc* G-H; 9. *Line* JF (K); 10. *Line* IK; 11. *Parallel to* IK *through* F (L, M); 12. *Circle* L-M *to complete*

DIVIDING A SEGMENT

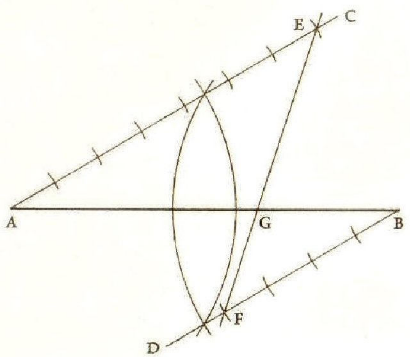
fair portions

A line segment may sometimes require precise division. Constructions 92 and 93 respectively show the division of a segment into equal parts and in a given proportion. In the latter construction any two lengths can be used. So, for example, instead of counting off equal divisions, lengths of $\sqrt{5}$ and $\sqrt{3}$ would divide AB in the ratio $\sqrt{5} : \sqrt{3}$.

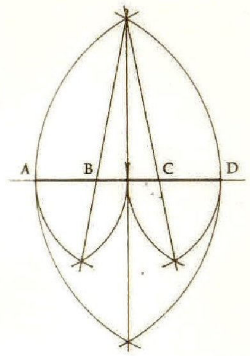
Constructions 95 and 96 are simplifications of constructions by Hana Hijazi. Constructions 97 and 98 use the same principle as constructions 92 and 93 but in a fixed context in which one parallel need not be drawn. Construction 99 relies on similar right triangles for the relationship shown – see if you can find them. Construction 100 combines constructions 94 and 99 to give $\frac{1}{3} \times \frac{1}{3} = \frac{1}{9}$, while construction 101 combines a $\frac{1}{4}$ division of AC with construction 99 to give $\frac{1}{16}$.



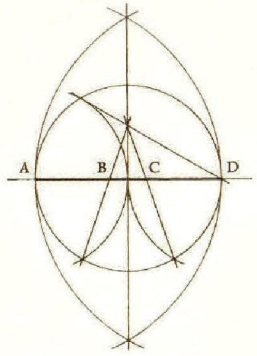
92. Dividing a segment into equal parts: 1. Arcs centres A, B (parallel lines AC, BD); 2. Mark off n equal partitions on AC, BD (in this case five); 3. Join partitions as shown to cut AB in n+1 equal divisions



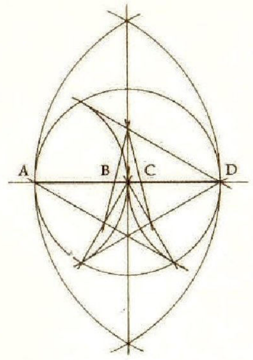
93. Dividing a segment into a given ratio: 1. Arcs centres A, B (parallel lines AC, BD); 2. Mark off lengths in given ratio on AC, BD, in this case 7 & 4 (E, F); 3. Line EF (G); AG:GB = AE:BF (in this case 7:4)



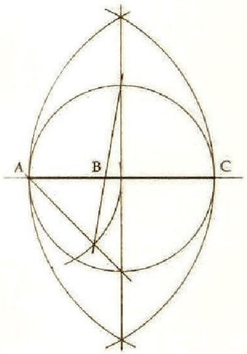
94. AB = BC = CD = $1/3$ AD



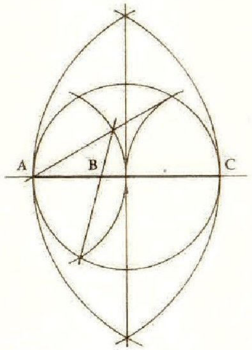
95. BC = $1/5$ AD



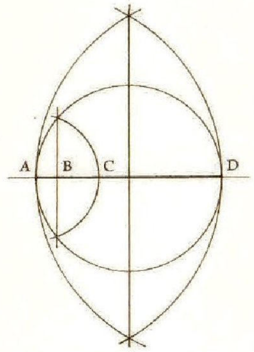
96. BC = $1/7$ AD



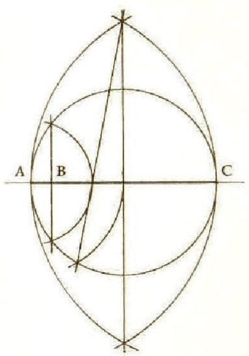
97. AB:BC = $1:\sqrt{2}$



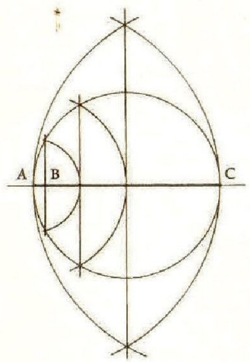
98. AB:BC = $1:\sqrt{3}$



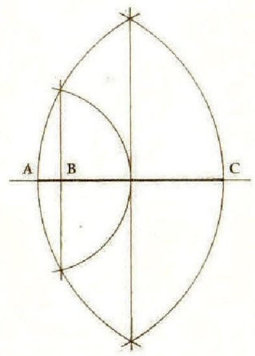
99. AB:AC = AC:AD



100. AB = $1/9$ AC



101. AB = $1/16$ AC

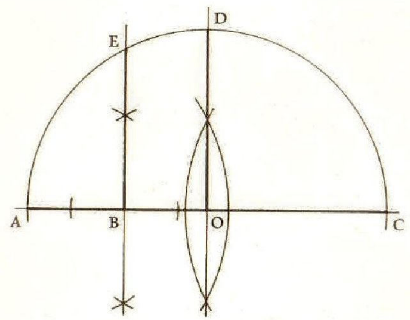


102. AB = $1/8$ AC

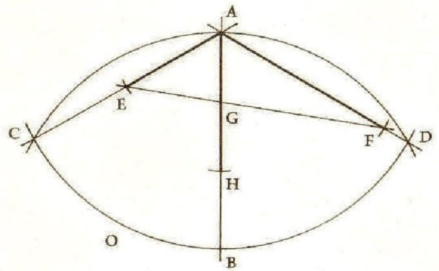
MEANS
produced by geometry

There are three essential means. The *arithmetic mean* is half the sum of two quantities, so the arithmetic mean of 2 and 4 is 3. The *geometric mean* is the quantity that is to the smaller as the larger is to it, so the geometric mean of 1 and 4 is 2, as 1 : 2 = 2 : 4. When three quantities are such that the first is to the third as the difference between the first and second is to the difference between the second and third then the second quantity is the *harmonic mean*, so the harmonic mean of 3 and 6 is 4, as 3 : 6 = 4 − 3 : 6 − 4 = 1 : 2.

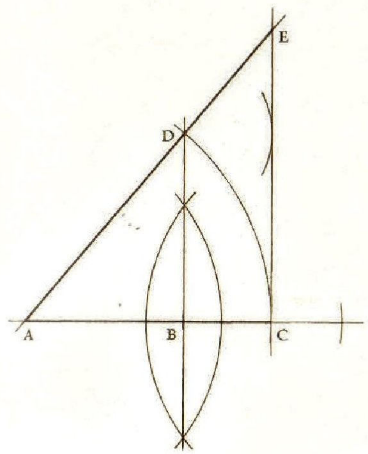
Construction 103 is from Pappus of Alexandria (d. *ca.* 350 AD) and shows the arithmetic and geometric means. Pappus also included the harmonic mean but for practical work construction 104, based on a diagram from Howard Eves (d. 2004), is better. Construction 105 finds a third length forming a geometric proportion with the first two. The geometric mean is very useful when working with areas as in constructions 106–108.



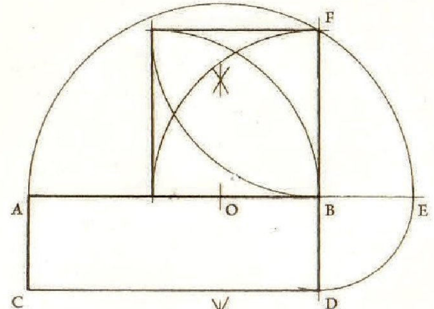
103. *Arithmetic and geometric means:*
1. *Mark two lengths* AB & BC *on a line;*
2. *Perp. bisector on* AC (O); 3. *Semicircle* O-AC (D); 4. *Perpendicular to* AC *on* B (E);
OD = *Arithmetic mean,* BE = *Geometric mean*



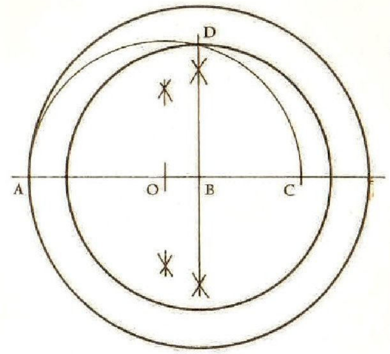
104. *Harmonic mean:*
1. *Arc centre* A *on a line* (B); 2. *Arc* B-A (C, D); 3. *Lines* AC, AD; 4. *Mark off two lengths* AE & AF *on* AC & AD;
5. *Line* EF (G); 6. *Arc* G-A (H);
AH = *Harmonic mean of* AE & AF



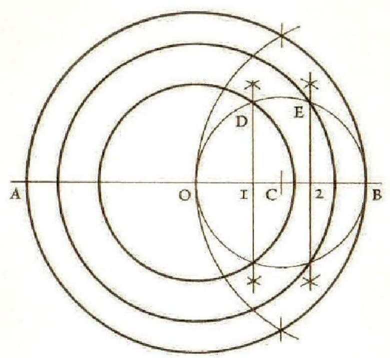
105. *Third proportional to two given lengths:*
1. *Mark off two lengths* AB & AC *on a line;*
2. *Perpendicular on* AC *through* B; 3. *Arc* A-C (D); 4. *Line* AD; 5. *Parallel to* BD *through* C (E); AB:AC = AC:AE



106. *Square equal in area to a rectangle:*
1. *Extend side* AB; 2. *Arc* B-D (E);
3. *Find midpoint of* AE (O); 4. *Semicircle* O-AE; 5. *Extend side* BD (F);
6. *Square on* BF *to complete*



107. *Two circles with areas in a given ratio:*
1. *Mark two lengths* AB & BC *in desired ratio on a line;* 2. *Circle* B-A
3. *Find midpoint of* AC (O);
4. *Semicircle* O-A; 5. *Perpendicular to* AC *through* B (D); 6. *Circle* B-D;
Area circle B-A:*Area circle* B-D = AB:BC



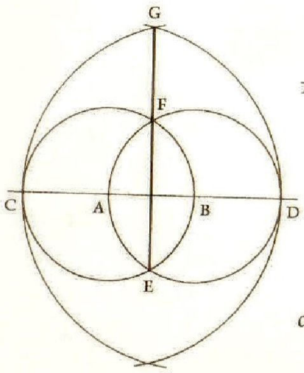
108. *Concentric circles forming equal areas:*
1. *Line through centre* O (A, B);
2. *Divide* OB *into equal parts, in this case three* (1, 2); 3. *Midpoint of* OB (C);
4. *Circle* C-OB; 5. *Perpendiculars to* OB *through divisions* (D, E); 6. *Circles* O-D, O-E;
Rings and central circle are equal in area

The Golden Section

der goldene snitch

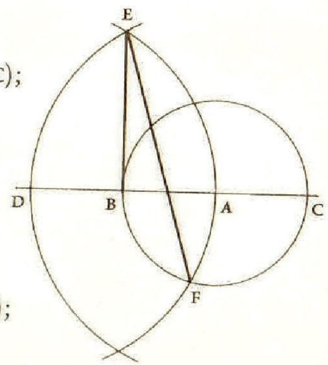
There is only one way to divide a given straight line segment such that the shorter part is to the longer as the longer part is to the whole segment, and this is when the longer part is the geometric mean of the shorter part and the whole segment. This was referred to by Euclid as *division in extreme and mean ratio*, but is better know today as the *golden section*, a name coined by Martin Ohm in about 1835. If the shorter of two lengths in the golden section is 1 then the longer is approximately 1.618034... often represented by the Greek letter Φ.

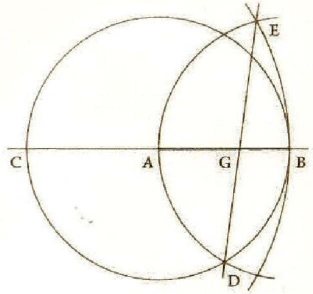
Constructions 109 to 113 are from papers by Kurt Hofstetter. They are all beautifully simple and show just how readily lengths in the golden section can be found with ruler and compass. They are particularly beautiful when drawn with full circles. Construction 114 is a practical version of a figure by George Odom, where A and B are the midpoints of two sides of the equilateral triangle DEF inscribed in the circle. This is closely related to construction 109. Construction 115 is the basis for many of the pentagon and decagon constructions on the following pages.



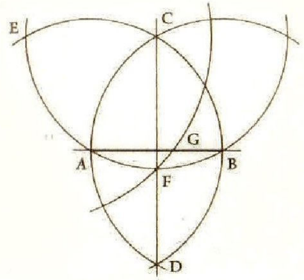
109. *Golden section:*
1. *Circle centre* A *on a line* (B, C);
2. *Circle* B-A (D, E, F);
3. *Arcs* A-D, B-C (G);
EF:FG = Φ:1

110. *Golden section:*
1. *Circle centre* A *on a line* (B, C); 2. *Arc radius* BC *centre* A (D); 3. *Arc* D-A (E, F); EF:EB = Φ:1

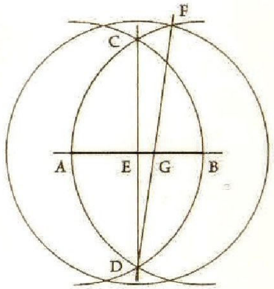




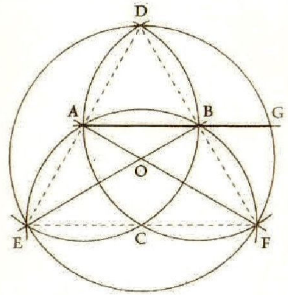
111. *Dividing a segment in the golden section:*
1. *Circle* A-B (C); 2. *Arc* B-A (D); 2. *Arc* C-B (E); 3. *Line* DE (G); AG:GB = Φ:1



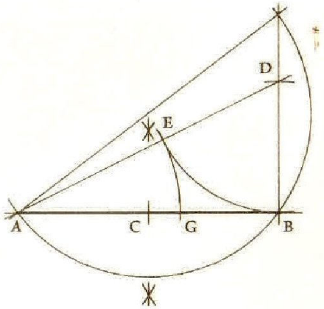
112. *Dividing a segment in the golden section:*
1. *Arcs* A-B, B-A (*line* CD); 2. *Arc* C-AB (E, F); 3. *Arc* E-F (G); AG:GB = Φ:1



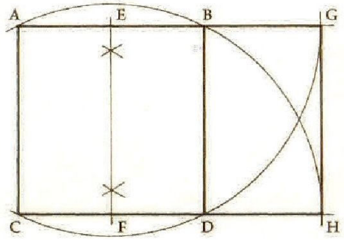
113. *Dividing a segment in the golden section:*
1. *Arcs* A-B, B-A (*line* CED); 2. *Circle radius* AB *centre* E (F); 3. *Line* DF (G); AG:GB = Φ:1



114. *Extending a segment in the golden section:*
1. *Arcs* A-B, B-A (C, D); 2. *Arc* C-AB (E, F); 3. *Lines* AF, BE (O); 4. *Circle* O-DEF; 5. *Extend* AB (G); AB:BG = Φ:1



115. *Dividing a segment in the golden section:*
1. *Find midpoint of* AB (C); 2. *Perp. to* AB *at* B; 3. *Arc* B-C (D); 4. *Line* AD; 5. *Arc* D-B (E); 6. *Arc* A-E (G); AG:GB = Φ:1



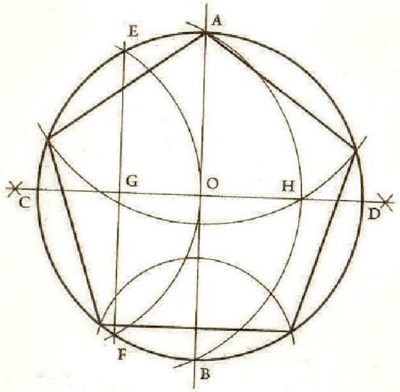
116. *Golden rectangle from a square:*
1. *Find midpoints of* AB, CD (E, F);
2. *Extend* AB, CD; 3. *Arcs* E-CD, F-AB (G, H); AG:AB = CH:CD = Φ:1

PENTAGONS & DECAGONS

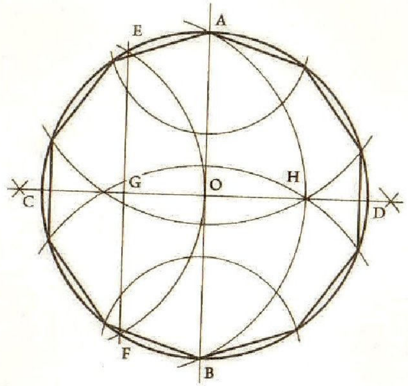
flowers and toes

Constructing a regular pentagon or decagon is less mathematically obvious than regular polygons such as triangles, squares, octagons, and dodecagons. The proportions needed are harder to find. Euclid gives a construction for a regular pentagon in Book Four of *The Elements*. However, as it is based on a long series of theorems it is not very practical, although it is mathematically elegant.

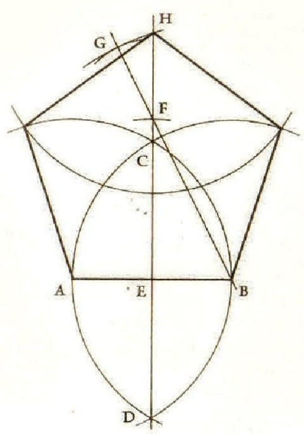
There are many simple practical constructions for regular pentagons and decagons. Construction 117 is a variation of the well-known construction in the *Almagest* of Ptolemy (d. *ca.* 168 AD) and construction 118 is an extension of this which produces the regular decagon. Constructions 119–122 all rely on simple applications of the golden section.



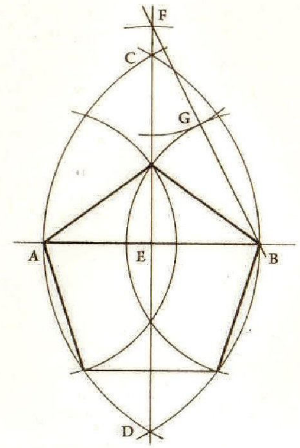
117. *Regular pentagon in a circle:* 1. *Line through centre* O (A, B); 2. *Perp. bisector on* AB (C, D); 3. *Arc* C-O (*line* EGF); 4. *Arc* G-AB (H); 5. *Arc* A-H; 6. *Arc radius* O-H *centre* B *& complete*



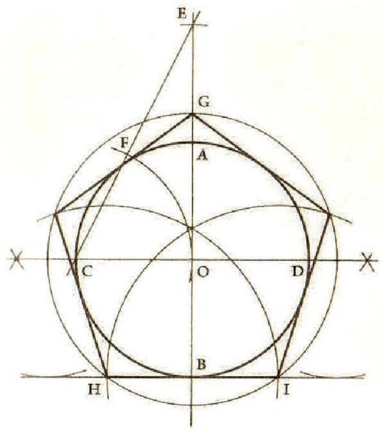
118. *Regular decagon in a circle:* 1. *to* 5. *as construction* 117; 6. *Arc* B-H; 7. *Arc radius* OH *center* A *& complete*; *Extra: Trace arcs radii* AH *&* OH *centres* C, D *to complete a regular icosagon (20 sides)*



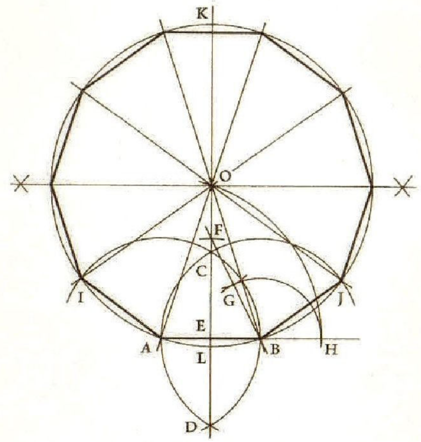
119. *Regular pentagon on a given side:* 1. *Arcs* A-B, B-A (*line* CED); 2. *Arc radius* AB *centre* E (F); 3. *Line* BF; 4. *Arc radius* AE *centre* F (G); 5. *Arc* B-G (H); *Arc radius* AB *centre* H *& complete*



120. *Regular pentagon on a given diagonal:* 1. *Arcs* A-B, B-A (*line* CED); 2. *Arc radius* AB *centre* E (F); 3. *Line* BF; 4. *Arc radius* AE *centre* F (G); 5. *Arcs radius* BG *centres* A, B *& complete*



121. *Regular pentagon around a circle:* 1. *Line through centre* O (A,B); 2. *Perp. bisector on* AB (C, D); 3. *Arc radius* AB *centre* O (E); 4. *Line* CE; 5. *Arc* C-O (F); 6. *Circle radius* EF *centre* O (G); 7. *Parallel to* CD *through* B (H, I); 8. *Arcs* H-I, I-H *& complete*



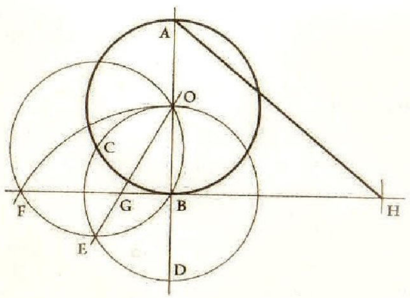
122. *Regular decagon on a given side:* 1. *Arcs* A-B, B-A (*line* CED); 2. *Arc radius* AB *centre* E (F); 3. *Line* FB; 4. *Arc radius* AE *centre* F (G); 5. *Extend side* AB; 6. *Arc* B-G (H); 7. *Arc* A-H (O); 8. *Circle* O-AB (I *to* L); 9. *Perp. bisector on* KL; 10. *Lines* IO, AO, BO, JO *& complete*

POSSIBILITIES
and impossibilities

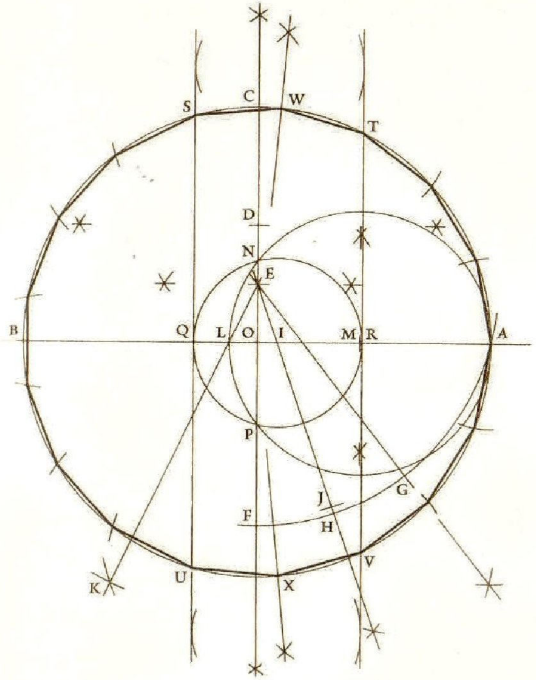
In 1637 René Descartes published *La Géométrie*, in which he introduced the algebraic study of geometric forms, and showed that ruler and compass constructions are equivalent to solving quadratic equations (equations of the form $ax^2 + bx + c = 0$, where a, b, and c are constants).

In 1801 Carl Friedrich Gauss proved in his *Disquisitiones Arithmeticæ* that any regular polygon with a Fermat prime number of sides can be constructed using ruler and compass (the only known Fermat primes are 3, 5, 17, 257, and 65,537), and likewise for multiples using each Fermat prime once, doubled any number of times, for example 3×5 or $2 \times 2 \times 17$ or $2 \times 5 \times 257$. Construction 124 is from H.W. Richmond in 1893.

In 1882 Ferdinand von Lindemann proved that π is a transcendental number, i.e., a number that is not a solution of any algebraic equation with whole number coefficients (which includes quadratic equations). It is therefore impossible, given a unit length, to construct with ruler and compass a length equal to π (3.14159...) or $\sqrt{\pi}$ (1.77245...), or more simply, to construct a square equal in area to a circle. Construction 123, from A. A. Kochansky in 1685, approximates π as 3.14153..., and construction 125, from E.W. Hobson in 1913, gives a value of 1.77247... for $\sqrt{\pi}$.

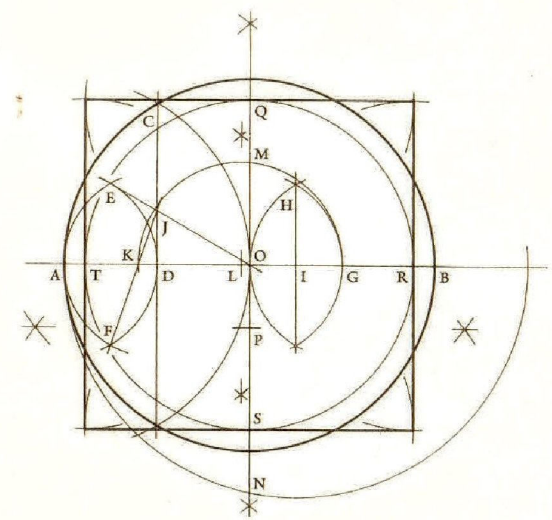


123. *Approximate quadrature:*
1. *Line through centre* O (A, B);
2. *Circle* B-O (C, D);
3. *Circle* C-OB (E);
4. *Arc* D-O (F);
5. *Line* FB; 6. *Line* OE (G);
7. *Arc radius* AD *centre* G (H);
AH:OA ≈ π:1



124. *Regular 17-gon in a circle:*
1. *Line through centre* O (A, B);
2. *Perp. bisector on* AB (C);
3. *Bisect* OC (D);
4. *Bisect* OD (E);
5. *Arc* E-A (F);
6. *Bisect* ∠AEF (G);
7. *Bisect* ∠GEF (H, I);
8. *Arc radius* OA=OB *centre* E (J);
9. *Arc radius* OA=OB *centre* J;
10. *Arc radius* BC=AC *centre* E (K);
11. *Line* EK (L);
12. *Bisect* AL (M);
13. *Circle* M-AL (N, P);
14. *Circle* I-NP (Q, R);
15. *Parallels to* CD *through* Q, R (S *to* V);
16. *Bisect* ∠SOT, ∠UOV (W, X);
17. *Use distances* ST=UV & SW=WT=UX=XV *to mark off remaining vertices & complete*

125. *Approximate quadrature:*
1. *Line through centre* O (A, B);
2. *Perp. bisector on* AB;
3. *Arc* A-O (*line* CD);
4. *Arcs* A-D, D-A (E, F);
5. *Arc radius* AD *centre* O (G);
6. *Arc* G-O (*line* HI);
7. *Line* EO (J);
8. *Line* FJ (K);
9. *Bisect* KG (L);
10. *Arc* L-GK (M);
11. *Arc* I-A (N);
MN:AO ≈ $\sqrt{\pi}$:1;
12. *Bisect* MN (P);
13. *Circle radius* PM=PN *centre* O (Q *to* T);
14. *Arcs* Q-O, R-O, S-O, T-O *& complete*

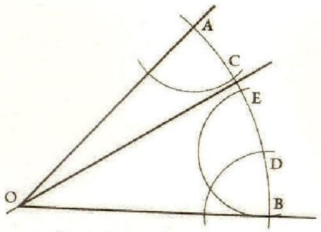


NEUSIS

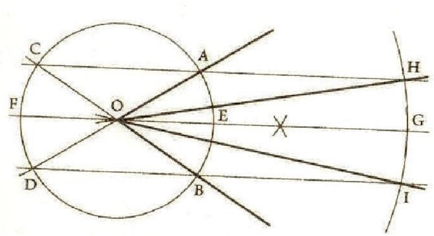
a useful trick

In 1837 Pierre Laurent Wantzel proved that it is impossible to find ruler and compass constructions which trisect an arbitrary angle, double the volume of a cube (find $\sqrt[3]{2}$), or make regular polygons other than those already proved possible by Gauss, as these problems are equivalent to solving equations with terms in x^3, x^4, x^5 or higher coefficients. Constructions 126 (by Jim Loy) and 127 are very accurate trisection approximations.

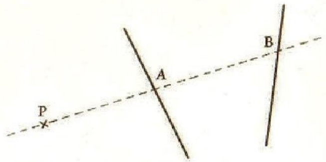
Some ruler and compass impossibilities can be performed exactly using *neusis* constructions. A given length is marked on the ruler or set as the opening of a pair of dividers, the ruler is placed through a given point and adjusted until the segment that lies between two given lines equals the given length (checked against the marks or dividers), and the line is drawn. Construction 128 is by Archimedes (d. *ca.* 212 BC), construction 129 is by John Conway, construction 130 is Isaac Newton's (d. 1727) solution for finding $\sqrt[3]{2}$, and construction 132, presented by Robin Hartshorne, is very similar to a construction by Hippocrates of Chios (d. *ca.* 410 BC).



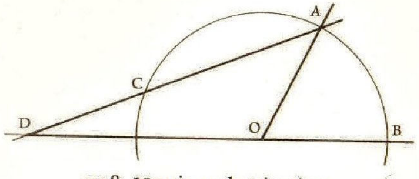
126. Estimated angle trisection:
1. Arc centre O (A, B); 2. Estimate trisection point C; 3. Arc radius AC centre B (D); 4. Arc D-B (E); 5. Estimate point 1/3 of the way along arc CE from C & complete



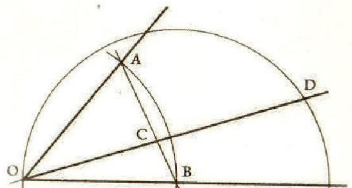
127. Approximate angle trisection:
1. Circle centre O (A, B); 2. Extend lines AO, BO (C, D); 3. Bisect ∠AOB (line EF); 4. Arc E-F (G); 5. Arc O-G; 6. Lines AC, BD (H, I); 7. Lines OH, OI



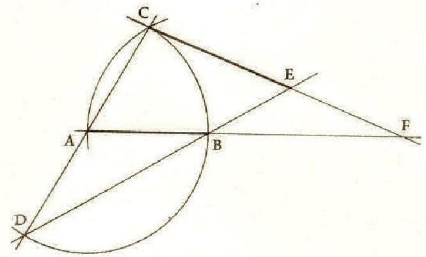
A neusis line insertion through point P so that AB equals a given distance



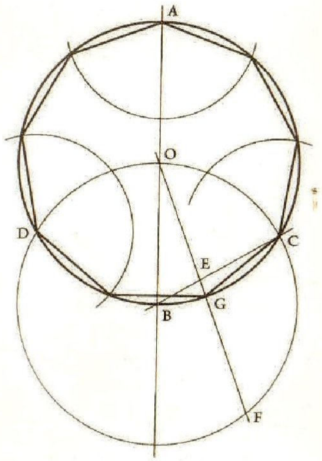
128. Neusis angle trisection:
1. Arc centre O (A, B); 2. Insert line ACD so that CD = OA; ∠ADB = 1/3 ∠AOB



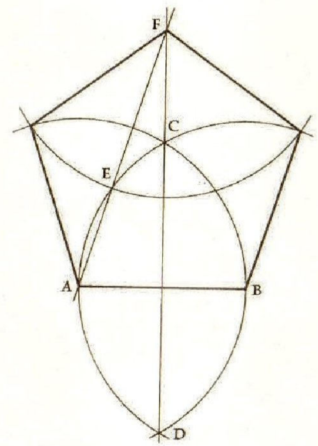
129. Neusis angle trisection in situ:
1. Arc centre O (A, B); 2. Semicircle B-O 3. Line AB; 4. Insert line OCD so that CD = OA; ∠DOB = 1/3 ∠AOB



130. Neusis cube root of two:
1. Line AB; 2. Arcs A-B, B-A (C); 3. Line AC (D); 4. Line DB; 5. Insert line CEF so that EF = AB; AB = 1, CE = $\sqrt[3]{2}$



131. Neusis regular enneagon in a circle:
1. Line through centre O (A, B); 2. Arc B-O (C, D); 3. Line BC; 4. Insert line OEF so that EF = OB (G); 5. Arcs radius CG centres C, D, A & complete



132. Neusis regular pentagon on a given side:
1. Arcs A-B, B-A (line CD); 2. Insert line AEF so that EF = AB; 3. Arc F-E & complete

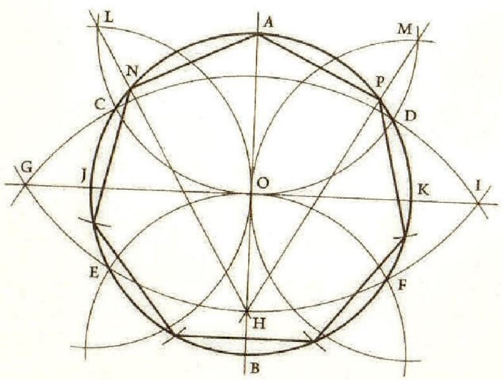
HEPTAGONS

secrets of seven

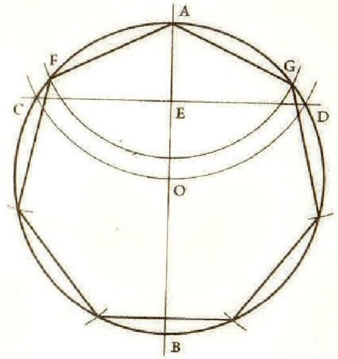
The heptagon is the regular polygon of least sides for which the ancient Greeks found no ruler and compass construction. We now know that ruler and compass construction of the regular heptagon is impossible, being equivalent to solving a cubic equation (one with terms in x^3). It can however be performed exactly using a *neusis* construction.

Construction 133 is by John Michell (d. 2009); it defines a side for an approximate regular heptagon subtending an angle of 51.444...° at the centre O (compared with 51.$\dot{4}$28571$\dot{1}$° for a regular heptagon).

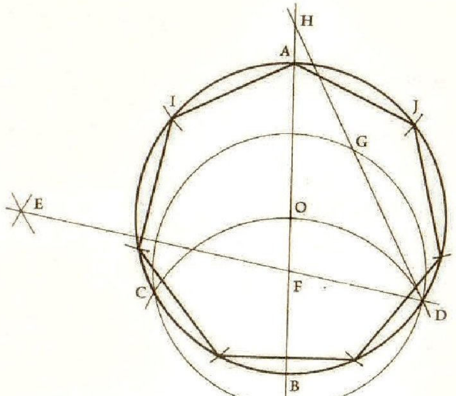
$\sqrt{3}/2$ has long been used as an approximation for the side of a regular heptagon inscribed in a unit circle, as it subtends an angle of 51.318...° at the centre. This approximation was used by Heron of Alexandria (d. 70 AD) in his *Metrica* and by Abu'l-Wafa' in his treatise for craftsmen, and constructions 134 and 136 both use this approximation. Construction 135 is François Viète's *neusis* heptagon from his 1593 *Supplementum Geometriæ*, and construction 137 is a *neusis* heptagon by Crockett Johnson (d. 1975).



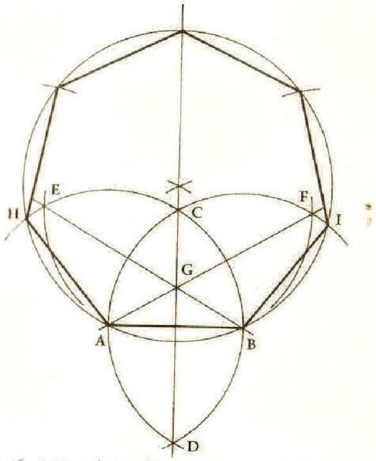
133. Approximate heptagon in a circle:
1. Line through centre O (A, B);
2. Arcs A-O, B-O (C to F);
3. Arcs B-CD, A-EF (G, H, I);
4. Line GI (J, K);
5. Arcs J-O, K-O (L, M);
6. Lines LH, MH (N, P);
7. Walk AN = AP around circumference from N, P & complete



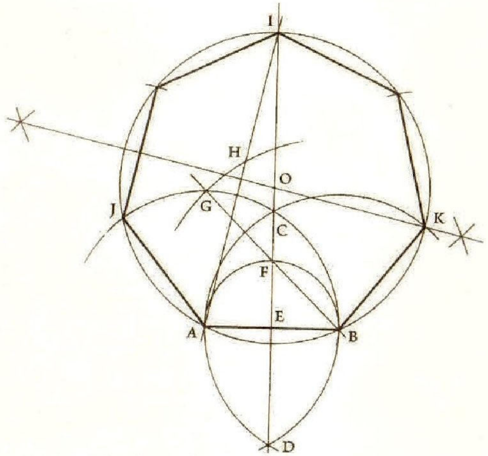
134. Approximate heptagon in a circle:
1. Line through centre O (A, B);
2. Arc A-O (Line CED);
3. Arc radius EC = ED centre A (F, G);
4. Walk EC = ED around circumference from F, G & complete



135. Neusis regular heptagon in a circle:
1. Line through centre O (A, B); 2. Arc B-O (C, D);
3. Arcs A-B, B-A (E); 4. Line DE (F); 5. Circle F-CD; 6. Insert line DGH so that GH = FD;
7. Arc radius OA centre H (I, J); 8. Walk AI = AJ around circumference from I, J & complete



136. Approximate heptagon on a given side:
1. Arcs A-B, B-A (line CD); 2. Arc C-AB (E, F); 3. Lines AF, BE (G);
4. Arcs radius GE centres A, B (O);
5. Circle O-AB (H, I);
6. Arcs H-A, I-B & complete



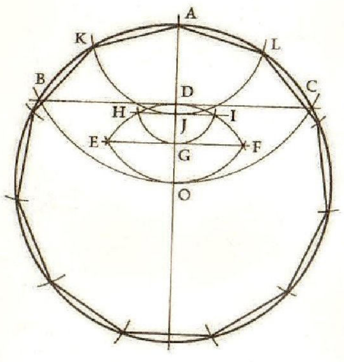
137. Neusis regular heptagon on a given side:
1. Arcs A-B, B-A (line CED); 2. Arc E-AB (F);
3. Line BF (G); 4. Arc B-G; 5. Insert line AHI so that HI = AB; 6. Perp. bisector on AI (O); 7. Circle O-AIB (J, K);
8. Arcs radius AB centres J, K & complete

Approximate Polygons

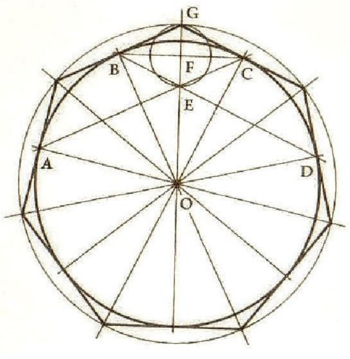
enneagons and beyond

A regular hendecagon cannot be constructed with ruler and compass or *neuseis*. Dürer recommended using $\frac{9}{16}$ of a circle's radius to approximate an inscribed hendecagon's side, subtending a central angle of $32.670...^\circ$ (correctly $32.\dot{7}\dot{2}^\circ$). Construction 138 is one way of doing this.

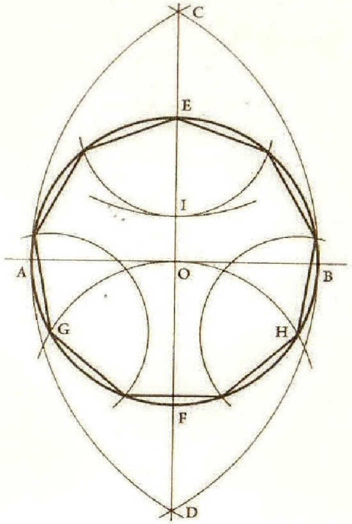
Construction 140 uses $1 + \sqrt{2} - \sqrt{3}$ in a unit circle as an approximation for a regular enneagon's side, subtending $39.886...^\circ$ at the centre O (correctly 40°). A regular triskaidecagon can be constructed using *neuseis* but this is tricky. Construction 141 uses a side approximation for the triskaidecagon that subtends an angle of $27.644...^\circ$ at the centre O (correctly $27.\dot{6}9230\dot{7}^\circ$). In construction 142 side AB subtends $40.208...^\circ$ at O (correctly 40°). In construction 143 side AB subtends $32.853...^\circ$ at O (correctly $32.\dot{7}\dot{2}^\circ$).



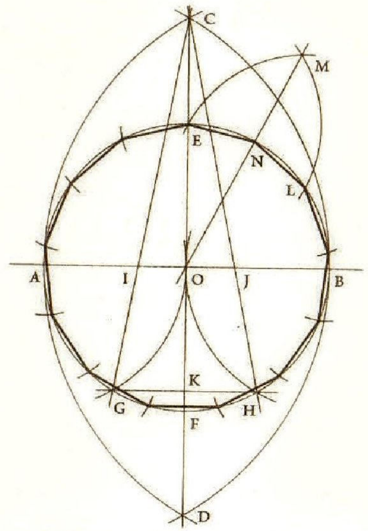
138. *Approximate hendecagon in a circle:* 1. *Line through centre* O (A); 2. *Arc* A-O (*line* BDC); 3. *Arcs* O-D, D-O (*line* EGF); 4. *Arc* D-G (*line* HJI); 5. *Arc* A-J (K, L); 6. *Walk radius* AK = AL *around circumference from* K, L *to complete*



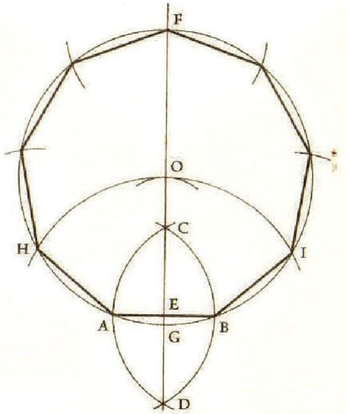
139. *General polygon around a circle:* 1. *Construct vertices of any inscribed polygon;* 2. *Lines from vertices through centre* O; 3. *Select vertices* A *to* D; 4. *Lines* AC, BD (E); 5. *Line* BC (F); 6. *Circle* F-E (G); 7. *Circle* O-G *to find vertices of circumscribed polygon & complete*



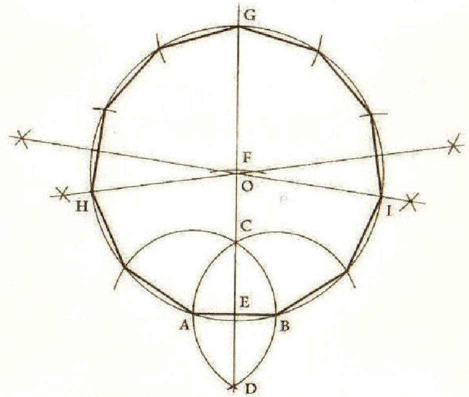
140. *Approximate enneagon in a circle:* 1. *Line through centre* O (A, B); 2. *Arcs* A-B, B-A (*line* CEFD); 3. *Arc* F-O (G, H); 4. *Arc radius* EA = EB *centre* C (I); 5. *Arcs radius* EI *centres* E, G, H *& complete*



141. *Approximate triskaidecagon in a circle:* 1. *Line through centre* O (A, B); 2. *Arcs* A-B, B-A (*line* CEFD); 3. *Arcs* A-O, B-O (G, H); 4. *Lines* CG, CH (I, J); 5. *Line* GH (K); 6. *Arc radius* KI = KJ *centre* E (L); 7. *Arc* L-E (*line* MNO); 8. *Walk* EN = EL *around circ. from* E, L *& complete*



142. *Approximate enneagon on a given side:* 1. *Arcs* A-B, B-A (*line* CED); 2. *Arc radius* EA = EB *centre* C (O); 3. *Circle* O-AB (F, G); 4. *Arc* G-O (H, I); 5. *Arcs radius* AB *centres* F, H, I *& complete*



143. *Approximate hendecagon on a given side:* 1. *Arcs* A-B, B-A (*line* CED); 2. *Arcs radius* CD *centres* E (F), F (G); 3. *Perp. bisector on* AG, BG (O); 4. *Circle* O-AGB (H, I); 5. *Arcs radius* AB *centres* H, G, I *& complete*

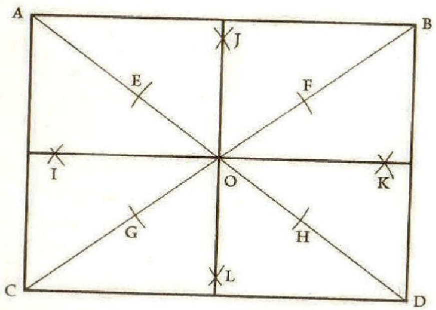
DIVIDING RECTANGLES

some crafty techniques

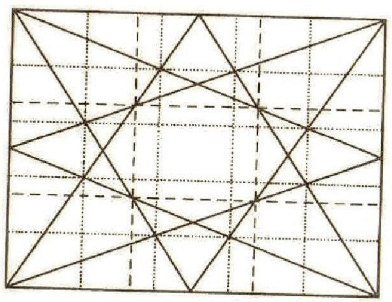
All the divisions shown here work for any rectangle (including a square).

Construction 145 is a fascinating and ancient device, sometimes known as the sand reckoner's diagram. The midpoints of each side are joined to the opposite two corners to create a lattice of vertices which can be used to divide the sides into 3, 4 and 5 equal parts. Especially useful in a square, the diagram has many other interesting properties and functions.

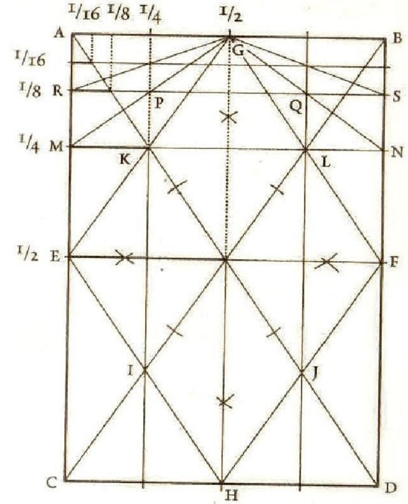
Constructions 146 and 147 are fairly efficient ways of subdividing rectangles into as many smaller rectangles as needed. ½, ¼, ⅛ ... and so on is a geometric series, in which the middle term of any three successive terms is the geometric mean of the other two. ½, ⅓, ¼ ... and so on is known as the harmonic series, in which the middle term of any three successive terms is the harmonic mean of the other two. Its fundamental role in harmony was discovered long ago by the Pythagoreans.



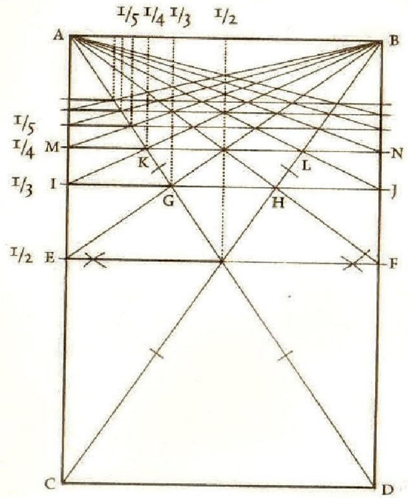
144. *Bisectors within a rectangle:*
1. *Lines* AD, BC (O); 2. *Arcs radius less than half* AO *centre* O (E, F, G, H); 3. *Arcs same radius centres* E, F, G, H (I, J, K, L); 4. *lines* IK, JL



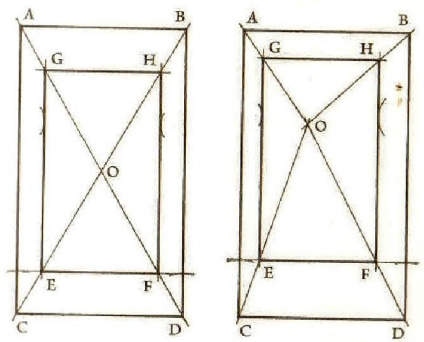
145. *'Sand Reckoner's' rectangle division:*
1. *Lines from each side's midpoint to the two opposite corners;* 2. *Intersections formed give division into 9 (dashed lines) and 25 (dotted lines) equal rectangles*



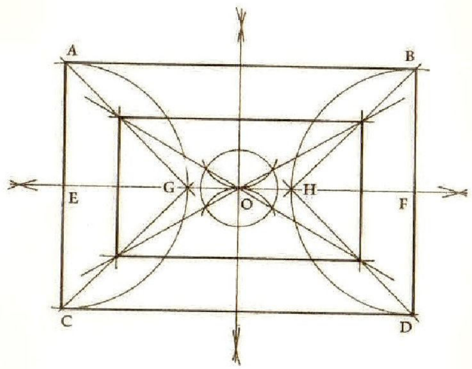
146. *Geometric division of a rectangle:*
1. *Lines* AD, BC; 2. *Find midpoints of* AB, CD, AC, BD (*line* EF, G, H); 3. *Lines* EG, EH, FG, FH (I, J, K, L); 4. *Lines* IK, JL; 5. *Line* KL (M, N); 6. *Lines* MG, NG (P, Q); *Line* PQ (R, S) ... *and so on*



147. *Harmonic division of a rectangle:*
1. *Lines* AD, BC; 2. *Find midpoints of* AC, BD (*line* EF); 3. *Lines* EB, FA (G, H); 4. *Line* GH (I, J); 5. *Lines* IB, JA (K, L); 6. *Line* KL (M, N) ... *and so on as needed*



148. *Similar rectangle within another:*
1. *For any point* O *within the rectangle, lines* AO, BO, CO, DO; 2. *Line* EF *parallel to* CD; 3. *Lines* EG, EH *parallel to* AC & BD; 4. *Line* GH



149. *Rectangle of given ratio within another:*
1. *Bisect* AB, CD & AC, BD (E, F, O); 2. *Semicircles* E-AC, F-BD (*lines* AG, CG, BH, DH); 3. *Diagonals of inner rectangle centred on* O (*in this case a* √3 *rectangle*) & *complete*

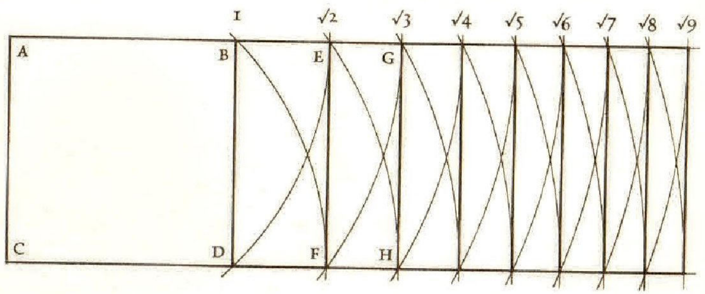
PROPORTIONAL RECTANGLES

secrets of ancient artisans

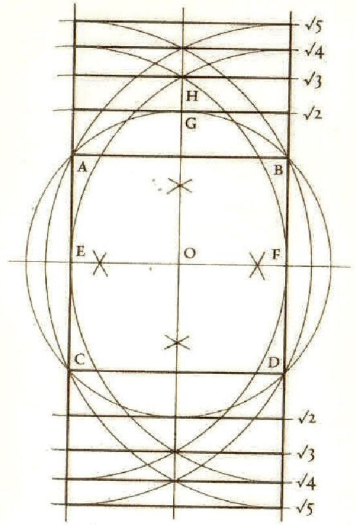
Jay Hambidge (d. 1924) proposed a reconstruction of ancient Greek art based on *dynamic symmetry* using proportional rectangles, namely rectangles with sides in the ratios of square roots of whole numbers to one; $\sqrt{2}:1$, $\sqrt{3}:1$, $\sqrt{4}:1$ and so on. Constructions 150 and 151 develop these root rectangles from a square while construction 152 develops them within a square. Hambidge also includes the golden rectangle in his system.

A rectangle inscribed on the short side of another so that both have sides in the same proportion is the larger rectangle's *reciprocal*. In a golden rectangle the reciprocal leaves a square. In a rectangle with sides of $\sqrt{n}:1$ the short side of the reciprocal rectangle divides the long side of the large rectangle into n parts, as shown in construction 153. The use of reciprocal rectangles and the right angled spirals derived from them is fundamental to Hambidge's system.

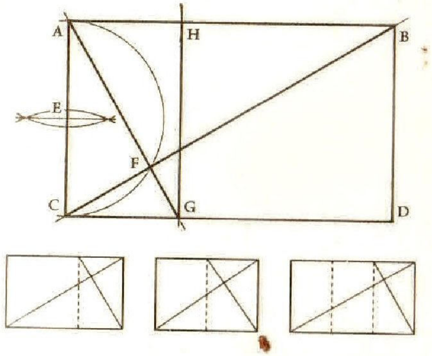
It may be worth noting here that any two opposite sides of a regular hexagon define a $\sqrt{3}$ rectangle.



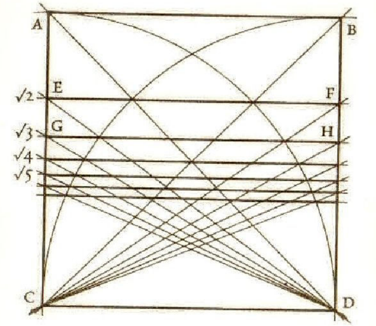
150. *Root rectangles from a square:*
1. *Extend AB, CD;* 2. *Arcs A-D, B-C (E, F & √2 rectangle);*
3. *Arcs A-F, B-E (G, H & √3 rectangle) ... and so on*



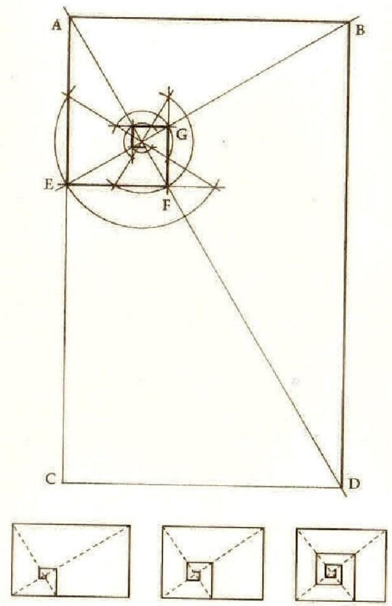
151. *Root rectangles from a square:*
1. *Extend AC, BD;* 2. *Bisect AB, CD;*
3. *Bisect AC, BD (E, F, O);* 4. *Circle O-A (G);*
5. *Arcs radius OG centres E, F (√2 rectangle);*
6. *Arcs E-F, F-E (H & √4 rectangle);*
7. *Arcs radius OH centres E, F (√3 rectangle);*
8. *Arcs E-BD, F-AC (√5 rectangle)*



153. *Diagonal & reciprocal of a rectangle:*
1. *Line BC;* 2. *Find midpoint of AC (E);* 3. *Semicircle E-AC (F);* 4. *Line AF (G);* 5. *Arc radius CG centre A (line GH)*



152. *Root rectangles in a square:*
1. *Lines AD, BC;* 2. *Arcs C-AD, D-BC (line EF & √2 rectangle);* 3. *Lines ED, FC (line GH & √3 rectangle) ... and so on*

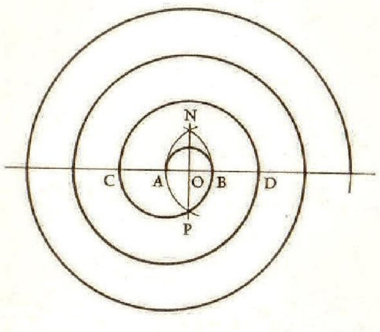


154. *Right-angled spiral from a rectangle:*
1. *Diagonal and reciprocal of rectangle (E);*
2. *Perp. to AC at E (line EF);* 3. *Perp. to EF at F (line FG) ... and so on*

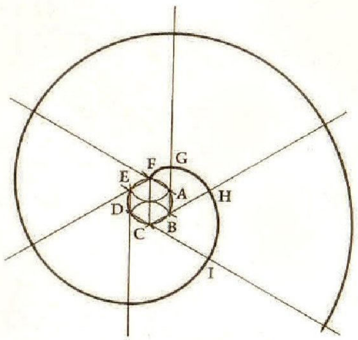
SPIRALS
and other perfect turnings

In his treatise *On Spirals* Archimedes examines the spiral formed by turning a line about one fixed end while moving a point along the line from this end at constant speed. These Archimedean spirals cannot be drawn exactly with ruler and compass. Construction 155 is a simple approximation while construction 156 is more accurate, but tricky.

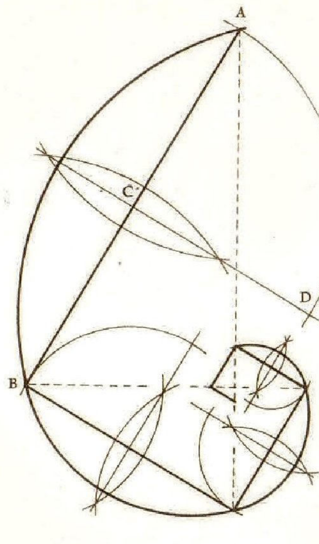
The logarithmic spiral was discovered by Descartes in 1638 and studied in particular by Jakob Bernoulli (d. 1705) who named it *spira mirabilis*. Logarithmic spirals are distinguished by the constant angle they form with radial lines from their centre; they can be more or less tightly wound depending on this angle. They can only be approximated with ruler and compass. A golden spiral is a special case of a logarithmic spiral with a particular relationship to the golden rectangle.

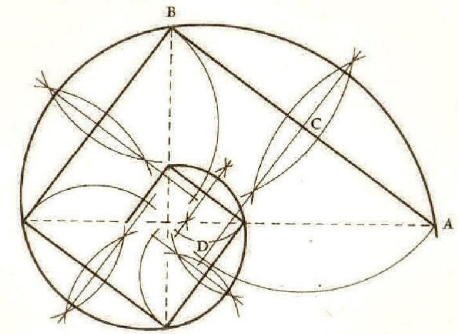


155. *Approximate Archimedean spiral:* 1. *Arc centre* A *on a line* (B); 2. *Arc* B-A (*line* NOP); 3. *Semicircle* O-AB; 4. *Semicircle* A-B (C); 5. *Semicircle* O-C (D); 6. *Semicircle* A-D ... *and so on*

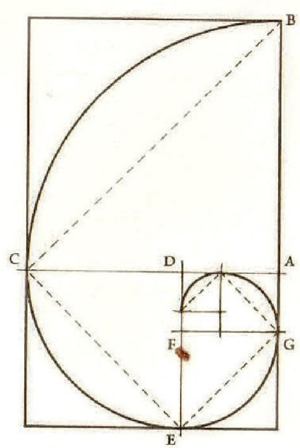


156. *Approximate Archimedean spiral:* 1. *Hexagon in a circle* (A *to* F); 2. *Extend hexagon's sides as shown;* 3. *Arc* A-F (G); 4. *Arc* B-G (H); 5. *Arc* C-H (I) ... *and so on*

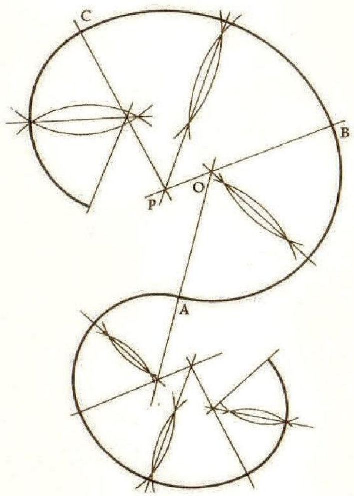




157. *Approximate logarithmic spiral:* 1. *Right angled spiral from any rectangle;* 2. *Perp. bisector on* AB (C); 3. *Arc* C-A (D); 4. *Arc* D-AB ... *repeat on other segments as needed*



158. *Approximate logarithmic golden spiral in a golden rectangle:* 1. *Mark off successive squares within the rectangle;* 2. *Arc* A-BC; 3. *Arc* D-CE; 4. *Arc* F-EG ... *and so on*



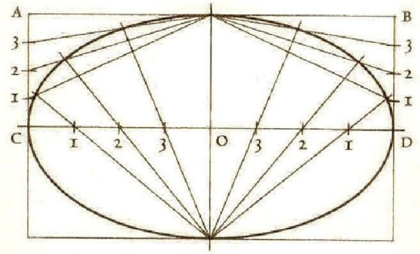
159. *Continuous curve:* 1. *Any arc* O-AB; 2. *Line* BO; 3. *For any subsequent point* C, *perp. bisector on* BC (P); 4. *Arc* P-BC ... *repeat as needed*

ELLIPSES & OVALS

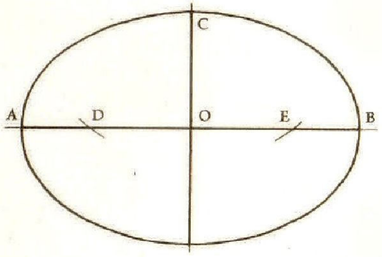
more heavenly curves

Slice a cone from one side to the other and the plane cross section will be a circle or an ellipse. Points on an ellipse of any aspect ratio (the ratio of major to minor axes) can be found as shown below. A rough continuous ellipse on given axes can be drawn using the Gardener's method.

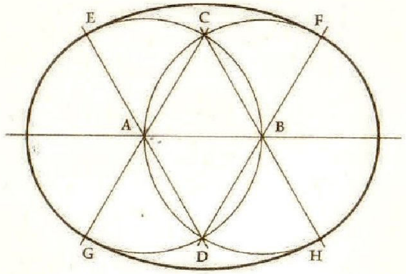
An oval is any curve that resembles an ellipse but is not a true ellipse. Ovals made from circular arcs can be constructed by ruler and compass, they were used in antiquity for buildings such as amphitheatres and became popular again in the Renaissance. Construction 162 is found in Sebastiano Serlio's (d. 1554) important treatise *Tutte l'Opere d'Architecttura*, its aspect ratio is approximately 1.323 : 1. Construction 163 from the architect Giacomo Vignola (d. 1573) is based on 3-4-5 right triangles (AOG for example). It has an aspect ratio of 3 : 2 and is remarkably close to a true ellipse. Construction 164 has an aspect ratio of $\sqrt{2}$: 1 and is also very close to a true ellipse. Construction 165 works for any aspect ratio AB : CD.



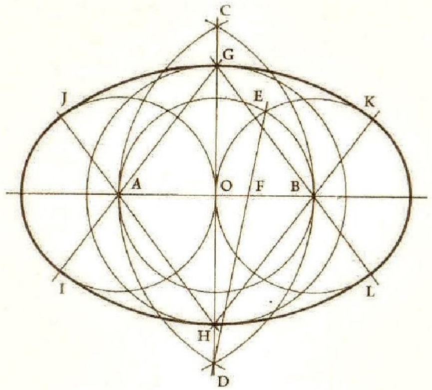
160. Points on an ellipse:
1. Divide AC, BD, CO, DO in a quartered rectangle into equal divisions; 2. Join the divisons as shown to find points on an ellipse



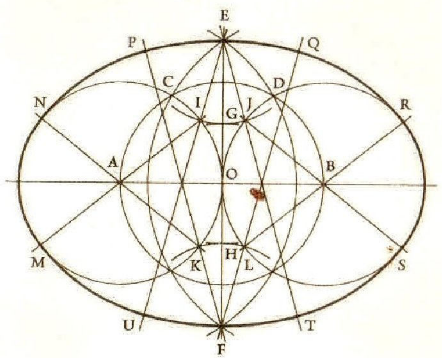
161. Gardener's method ellipse:
1. Arc radius OA centre C (D, E); 2. Place pins at A, B and tie string tautly between them; 3. Move pins to D, E; 4. Pull string taut with pen to trace curve



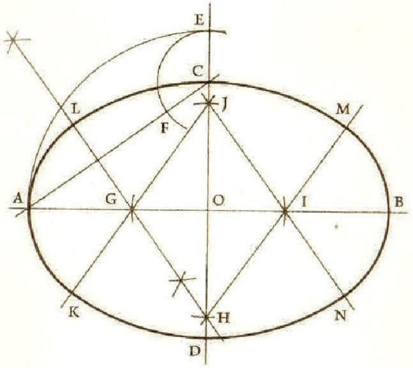
162. Simple fixed aspect ratio four-arc oval:
1. Circle centre A on a line (B); 2. Circle B-A (C, D); 3. Lines DA, DB, CB, CA (E, F, G, H); 4. Arcs D-EF, C-GH



163. Fixed aspect ratio four-arc oval:
1. Circle centre O on a line (A, B); 2. Arcs A-B, B-A (line CD); 3. Circles A-O and B-O (E); 4. Line ED (F); 5. Circle radius AF centre O (G, H); 6. Lines GA, HA, HB, GB (I, J, K, L); 7. Arcs A-IJ, H-JK, B-KL, G-LI



164. Fixed aspect ratio six-arc oval:
1. Circle centre O on a line (A, B); 2. Circles A-O and B-O (C, D); 3. Arcs B-C, A-D (line EF); 4. Arcs radius AB centres F, E (G, H); 5. Arcs E-G, F-H (I, J, K, L); 6. Lines IA, KA, LB, JB (M, N, R, S); 7. Lines FK, FL, EI, EJ; 8. Arcs F-E, E-F (P, Q, T, U); 9. Arcs as needed centres A, K, F, L, B, J, E, I through M, N, P, Q, R, S, T, U

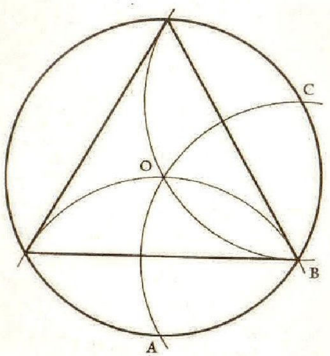


165. Flexible aspect ratio four-arc oval:
1. Extend CD if needed; 2. Arc O-A (E); 3. Line AC; 4. Arc C-E (F); 5. Perp. bisector on AF (G, H); 6. Arc O-G (I); 7. Arc O-H (J); 8. Lines JG, HG, HI, JI; 9. Arcs G-A & I-B (K, L, M, N); 10. Arcs H-LCM & J-NDK

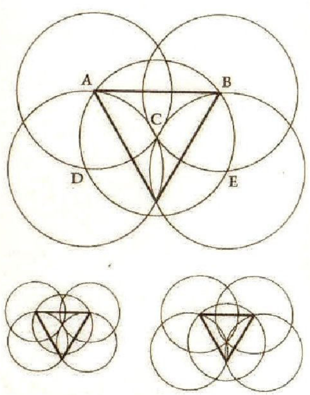
Rusty Compass
just one circle

The mathematician best known for rusty compass constructions is Abu'l-Wafa', although earlier works such as the *Mathematical Collection* of Pappus of Alexandria also contain examples. Like Euclid's *collapsible compass*, the rusty compass seems to be a mathematical abstraction rather than a reflection of an actual tool. No respectable artisan lets their tools rust solid. But practical geometers also know that each change of compass opening can be a source of error.

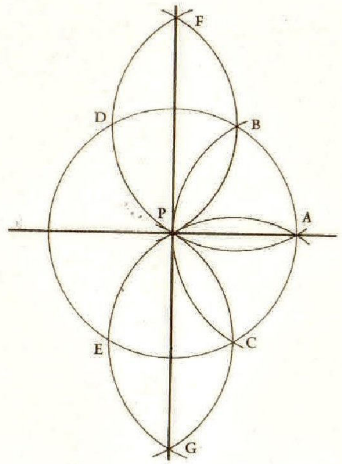
The remarkable construction 167 was noted by Dan Pedoe (d. 1998) in an anonymous student's sketch. Construction 170 is based on Abu'l-Wafa's work and construction 171 is from Kurt Hofstetter. Other rusty compass constructions in this book include 13 (from Abu'l Wafa'), 34, and 113. Constructions 166 to 168 are also compass only (*see too overleaf*).



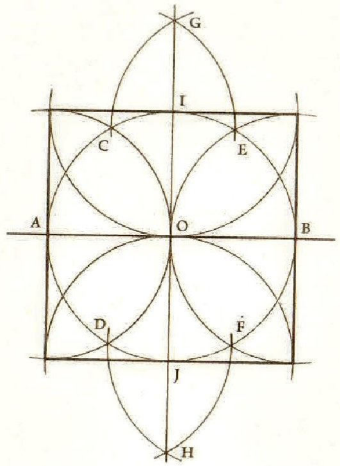
166. *Equilateral triangle in a circle:* 1. *Circle centre* O; 2. *Arc centre* A *on circumference* (B); 3. *Arc centre* B (C); 4. *Arc centre* C & *complete*



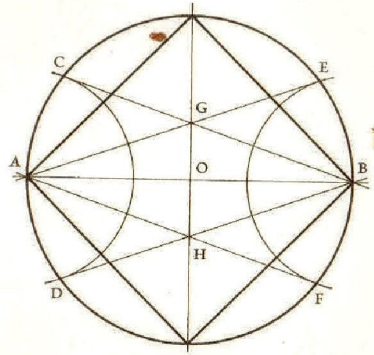
167. *Equilateral triangle on a given side* AB: 1. *Circles centres* A, B (C); 2. *Circle centre* C (D, E); 3. *Circles centres* D, E & *complete*



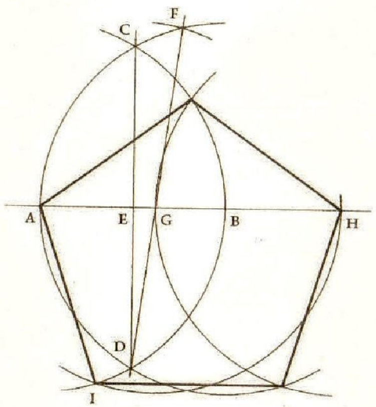
168. *Perpendicular through point* P *on a line:*
1. *Circle centre* P (A);
2. *Arc centre* A (B, C);
3. *Arcs centres* B, C (D, E);
4. *Arcs centres* D, E (F, G);
Line FG *is perpendicular to* AP



169. *Square set orthogonally on a line:*
1. *Circle centre* O (A, B);
2. *Arcs centres* A, B (C, D, E, F);
3. *Arcs centres* C, D, E, F (*line* GIJH);
4. *Arcs centres* I, J & *complete*



170. *Square in a given circle:*
1. *Line through centre* O (A, B);
2. *Arcs centres* A, B (C, D, E, F);
3. *Lines* CB, DB, EA, FA (G, H);
4. *Line* GH & *complete*



171. *Regular pentagon:*
1. *Arc centre* A *on a line* (B); 2. *Arc centre* B (*line* CED); 3. *Arc centre* E (*line* FGD); 4. *Arc centre* G (H, I); 5. *Arc centre* H & *complete*

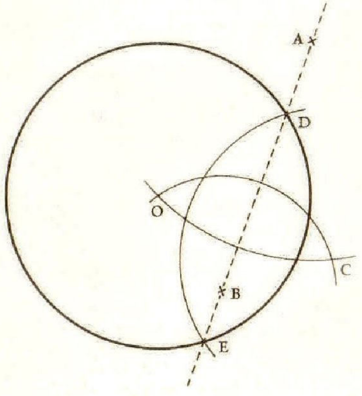
COMPASS ALONE

or ruler alone

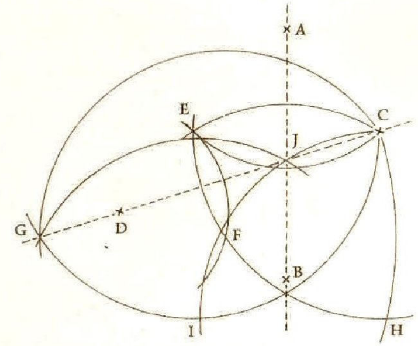
In 1672 Georg Mohr published a proof that all ruler and compass constructions can be completed using compass alone (assuming a straight line to be constructed if two or more points on it are found). The same result was found independently in 1797 by Lorenzo Mascheroni.

The proof of the Mohr-Mascheroni theorem rests on showing that the intersections of two lines, or a line and a circle can be found by compass alone (*two examples are shown below*). Construction 174 is Mascheroni's solution to finding the centre of a circle. Construction 175 is Fitch Cheney's solution to Napoleon's problem (purportedly proposed by the Emperor himself). Construction 177 is by Michel Bataille.

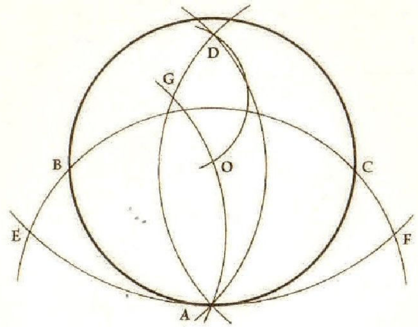
In 1833 Jakob Steiner proved that all ruler and compass constructions can be completed using ruler only, as long as an initial circle is given.



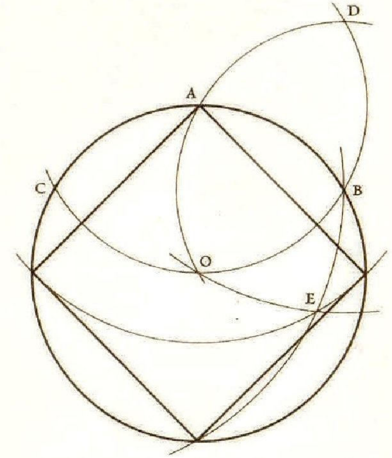
172. Compass only intersection of circle and line: 1. Arc A-O; 2. Arc B-O (C); 3. Arc radius equal to circle's radius, centre C (D, E)



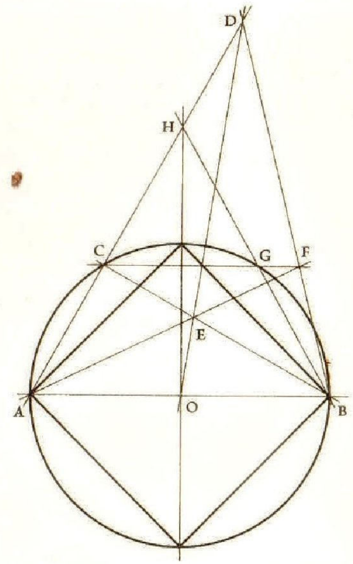
173. Compass only intersection of two lines: 1. Arc A-C; 2. Arc B-C (E); 3. Arc D-E; 4. Arc C-E (F); 5. Arc E-C; 6. Arc F-C (G); 7. Arc G-C (H); 8. Arc H-C (I); 9. Arc I-G (J)



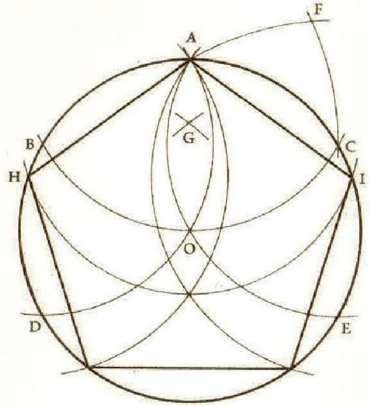
174. Centre of a circle using compass only: 1. Arc any suitable radius, centre A (B, C); 2. Arcs B-A, C-A (D); 3. Arc D-A (E, F); Arc E-A (G); 4. Arc G-D to find centre O



175. Square in a circle using compass only: 1. Arc A-O (B, C); 2. Arc B-AO (D); 3. Arcs C-B, D-O (E); 4. Arc A-E & complete

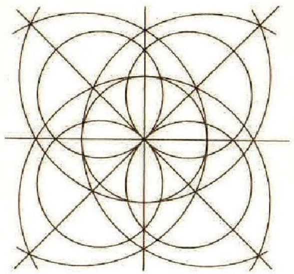


176. Square in a circle using ruler only: 1. Line through centre O (A, B); 2. Line AC for any suitable C; 3. Line through B (D); 4. Line OD; 5. Line BC (E); 6. Line AE (F); 7. Line CF (G); 8. Line BG (H); 9. Line HO & complete

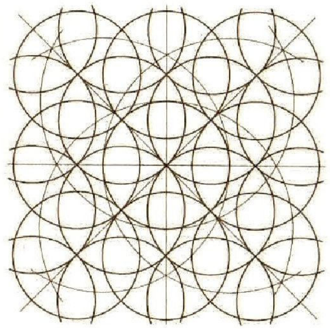


177. Pentagon in a circle using compass only: 1. Arc A-O (B, C); 2. Arcs B-A, C-A (D, E); 3. Arcs B-C, E-A (F); 4. Arcs radius OF centres D, E (G); 5. Arc radius OA centre G (H, I); 6. Arcs H-A, I-A & complete

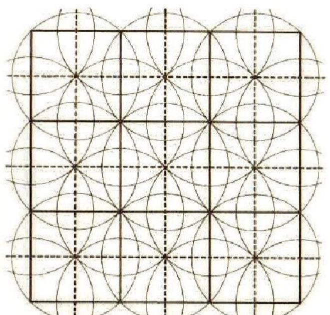
Appendix - Grid Constructions



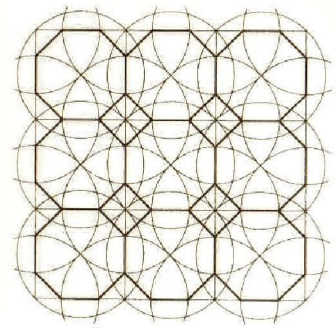
A square grid can be made by subdividing a square, or it can be constructed outwards starting as above,



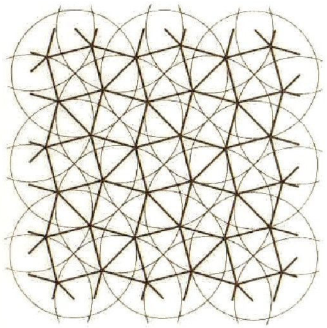
and continuing to give a pattern of overlapping circles,



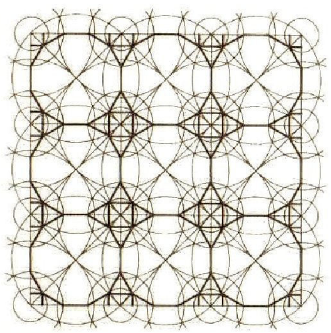
which defines a square grid and its dual (dashed).



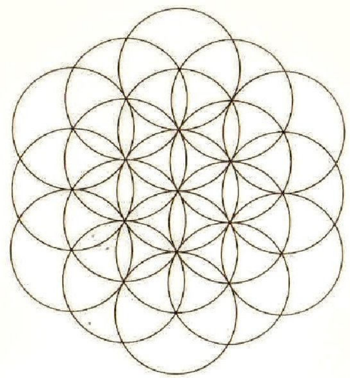
Each square also contains construction 55.



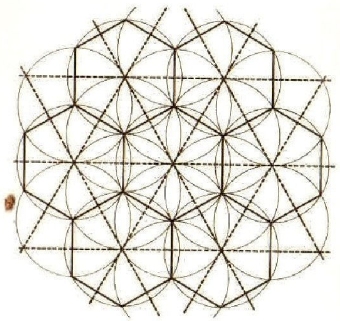
The circles also define the grid above.



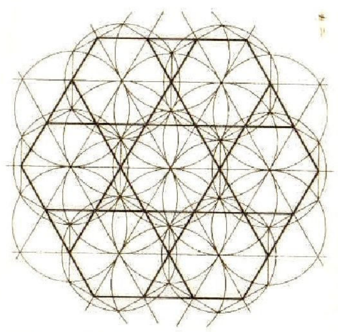
Construction 53 can be added to each square.



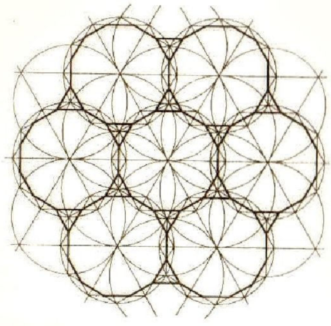
Construction 3, drawn with complete circles, can be continued to give the pattern above,



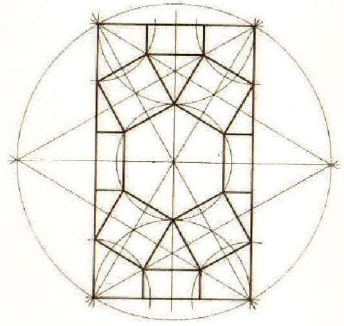
defining dual hexagonal and triangular grids.



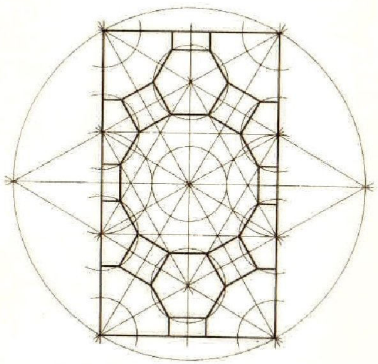
Join the midpoints of each hexagon's sides to form this grid of triangles and hexagons.



Add hexagons rotated by 90 degrees to each circle to define a grid of dodecagons and triangles.

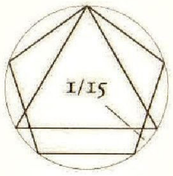


The construction above gives a rectangular section of a grid of triangles, squares & hexagons.

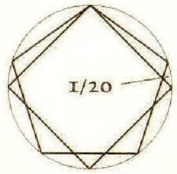


A related construction gives a grid of squares, hexagons and dodecagons.

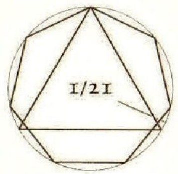
APPENDIX - POLYGON COMBINATIONS



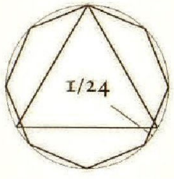
$15 = 3 \times 5$



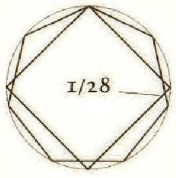
$20 = 4 \times 5$



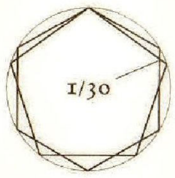
$21 = 3 \times 7$



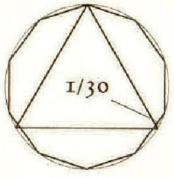
$24 = 3 \times 8$



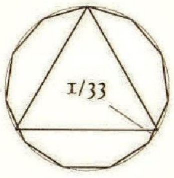
$28 = 4 \times 7$



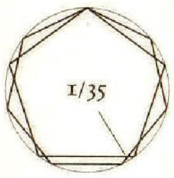
$30 = 5 \times 6$



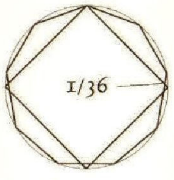
$30 = 3 \times 10$



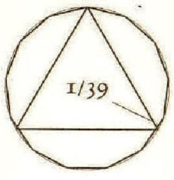
$33 = 3 \times 11$



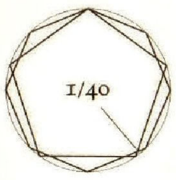
$35 = 5 \times 7$



$36 = 9 \times 4$



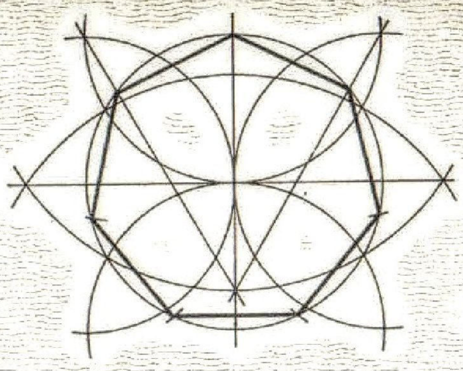
$39 = 3 \times 13$

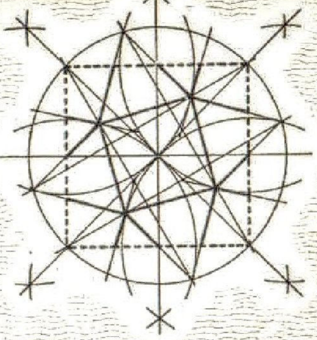


$40 = 8 \times 5$

Euclid gives a construction of the regular 15-gon that uses an equilateral triangle and a regular pentagon inscribed in the same circle to find the length of the 15-gon's side (top left).

This can be extrapolated to other regular polygons. For practical work the missing vertices should be marked off by using the vertices of the two defining polygons as staging posts.





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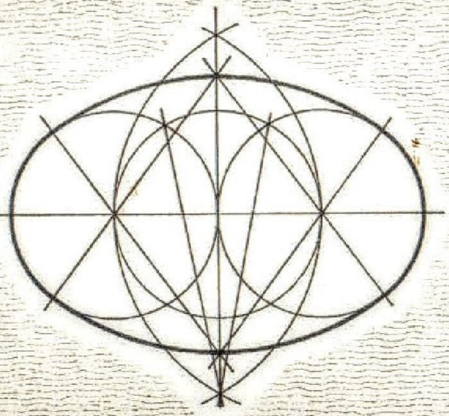

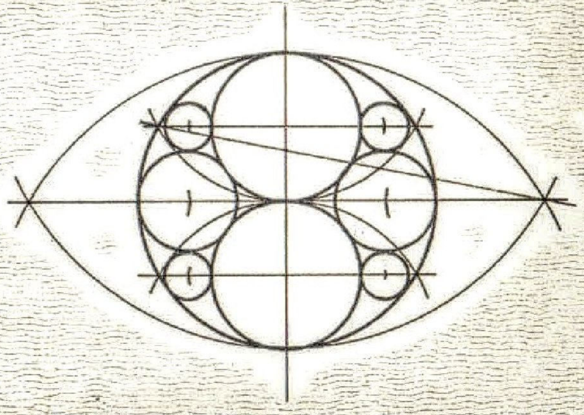

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